VC dimension with half guards

Erik Krohn*[1] and Alex Pahlow†[2]

[1]University of Wisconsin-Oshkosh
[2]University of Wisconsin-Madison

Abstract

In this paper, we show that the VC dimension of half-guarding a terrain is exactly 2 or 3, depending on certain assumptions. We also show that the VC dimension of half-guarding a monotone polygon is exactly 4.

1 Introduction

A full guard is a guard that can see 360°. In our paper, we define a *half guard* as a guard that sees 180° and only sees to the right. VC dimension is a measure of the complexity of some set system. It has been studied by researchers for many variants of the art gallery problem. Guarding simple polygons with full guards has a VC dimension between 6 and 14 [1]. Guarding monotone polygons (simple polygons) with full guards where guards are limited to being on the boundary of the polygon was shown to be exactly 6 in both types of polygons [2, 3]. The structure half guards add to the art gallery problem is interesting because the difference, as compared to, full guards, is not trivial. For example, convex polygons have a VC dimension of 1 with half guards despite having a VC dimension of 0 with full guards. Monotone polygons, where all guards and viewpoints are located on the boundary, have a VC dimension of 4 with half guards despite having a VC dimension of 6 with full guards.

A set of guards G in P is *shattered* if for every $G_s \subseteq G$, there exists a point that is seen by the guards in G_s and by no guards in $G \setminus G_s$. With half guarding, we show that the VC dimension is exactly 4. The terrain guarding problem with full guards has a VC dimension of exactly 4 [4]. With half guarding, we show that the VC dimension is exactly 2 or 3, depending on certain assumptions.

Notation: Let $p < q$ mean that point p is to the left of q, i.e. the x coordinate of $p.x < q.x$. With half guarding a polygon (resp. terrain), a point p sees a point q if the line segment connecting p and q does not go outside of the polygon (resp. below the terrain) and $p.x \leq q.x$. Let p and q be two points such that $p.x < q.x$, then $[p, q)$ denotes every point in the polygon between p and q (including the vertical line containing p but excluding the vertical line containing q). Let l be the leftmost point of the polygon and let r be the rightmost point of the polygon. The *ceiling (resp. floor)* denotes every boundary point in $[l, r]$ as we travel clockwise (resp. counterclockwise) from l to r. We define *viewpoint* as a point that is exactly seen by a subset of the guards. For example, the viewpoint $vp(AC)$ is a point in the polygon that is seen by guards A and C but is not seen by any other guards.

2 VC dimension of terrains

We start by discussing the VC dimension of terrains with regards to half guards. The VC dimension of a terrain with regards to half guards depends on if a point on the terrain can be considered both a guard and a viewpoint. If guards and viewpoints must be disjoint, then the VC dimension is 2. If a point on the terrain can be both a guard and a viewpoint, then the VC dimension is 3. Figure 1 shows an example of a terrain being shattered with 2 guards. We use the standard *order claim* without proof.

Claim: Let A, B, C, D be 4 points on a terrain with $A.x < B.x < C.x < D.x$. If A sees C and B sees D, then A must see D.

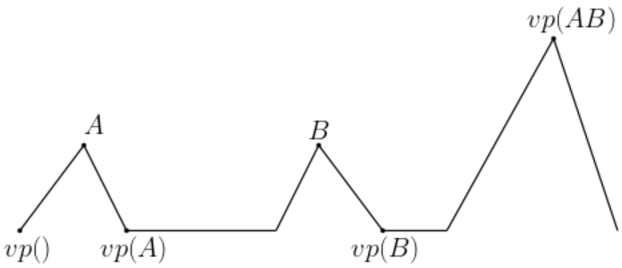

Figure 1: A terrain shattered by 2 half guards.

Theorem 1 *If a terrain guarding problem does (resp. does not) allow a guard and a viewpoint to be the same point, then the VC dimension of a terrain is exactly 3 (resp. 2).*

*Email: krohne@uwosh.edu
†Email: apahlow22@alumni.uwosh.edu

Proof. We will first consider the case where guards and viewpoints cannot be located at the same point. Let A, B and C be guards such that $A.x < B.x < C.x$. Assuming that a guard and viewpoint cannot be the same point, the viewpoints that are seen by C must be strictly to the right of C. It follows that $B.x < C.x < vp(BC).x$ and $B.x < C.x < vp(AC).x$. If $vp(BC).x < vp(AC).x$, then we have $B.x < C.x < vp(BC).x < vp(AC).x$. By the order claim, B sees $vp(AC)$, a contradiction. If $vp(AC).x < vp(BC).x$, then $A.x < B.x < vp(AC).x < vp(BC).x$. By the order claim, A sees $vp(BC)$, a contradiction.

Next we consider the VC dimension of terrains where a guard and a viewpoint can be at the same point. In this case, the VC dimension is 3. We achieve a lower bound of 3 by giving an example of a terrain shattering 3 guards in Figure 2.

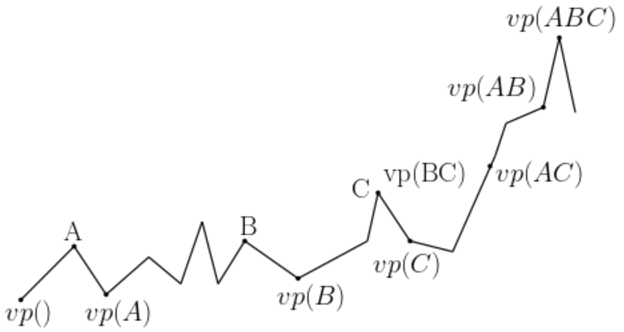

Figure 2: A terrain shattered by 3 half guards. In this example, C and $vp(BC)$ are the same point.

We will show that it is impossible for such a terrain to have a VC dimension of 4. Let A, B, C, D be the guards of this polygon with $A.x \leq B.x \leq C.x \leq D.x$. Consider the following cases:

1. If the viewpoint $vp(AC).x < vp(BD)$, then $A.x < B.x < vp(AC).x < vp(BD).x$. By the order claim using $A, B, vp(AC), vp(BD)$, A sees $vp(BD)$.

2. If the viewpoint $vp(BD).x < vp(AC)$, then $B.x < C.x < vp(BD).x < vp(AC).x$. By the order claim using $B, C, vp(BD), vp(AC)$, B sees $vp(AC)$.

$\square$

3 VC dimension of monotone polygons

We show that the VC dimension of half guarding a monotone polygon is exactly 4. We obtain the lower bound for monotone polygons by giving an example of a monotone polygon being shattered by 4 guards as seen in Figure 3. We now show that the 5 guards cannot be shattered with a case analysis. A few cases are shown in the paper with the remaining ones omitted due to lack of space. We use the following lemma:

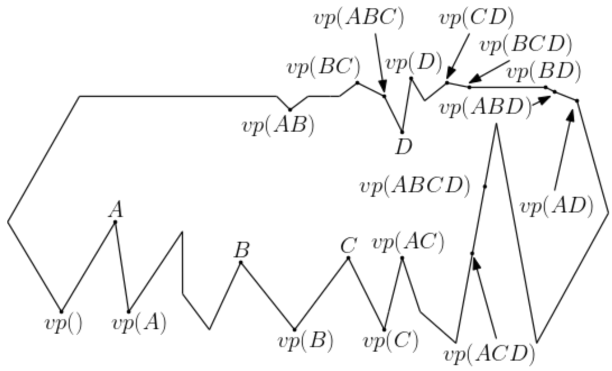

Figure 3: Polygon shattered by 4 half guards.

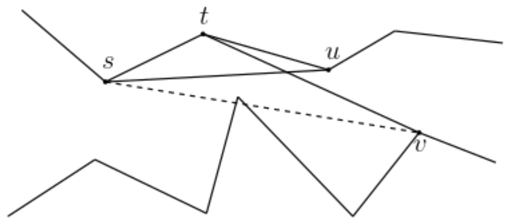

Figure 4: Lemma 2 where s, t and u are on the ceiling.

Lemma 2 *Let $s < t < u < v$ where s, t, u are on the same side of the polygon, s sees u, t sees v, and s does not see v. The opposite side of the polygon must block s from seeing v.*

Proof. W.l.o.g., assume s, t and u are on the ceiling. If a point p' on the ceiling is used to block s from v such that $s.x < p'.x < u.x$, then s is blocked from u. If a point p' on the ceiling is used to block s from v such that $t.x < p'.x < v.x$, then t is blocked from v. If the ceiling wraps underneath v to block s from v, then the polygon is not monotone. Therefore, if s does not see v, the floor must block it. $\square$

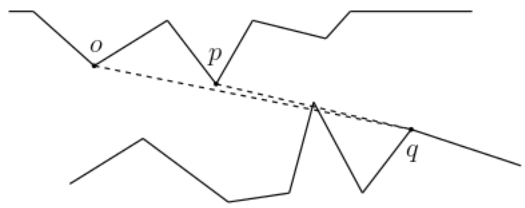

Figure 5: Lemma 3 where p is on the ceiling and the floor blocks p from q.

Lemma 3 *Let p and q be two points in the polygon and let $p < q$. If p is blocked from q using the side opposite p, then no point in $[l, p]$ can see q.*

Proof. W.l.o.g, assume that p is on the ceiling and the floor is blocking p from q. Let o be some point to the left of p. The $\overrightarrow{oq}$ ray lies in between the $\overrightarrow{pq}$ ray and the floor. If this were not the case, then p would have blocked o from q. If the floor blocks p from q, the $\overrightarrow{oq}$ ray must also go through the floor and therefore, q must also be blocked from o. $\square$

Lemma 4 *Let $s < t < u$, where t and u are on opposite sides of the polygon, s sees u, and t does not see u. It must be that t cannot see any point in $[u, r]$.*

Proof. Assume, w.l.o.g., that t is on the floor. Note that t cannot be blocked from u using the ceiling since by Lemma 3, s would not see u. Thus, t must be blocked from u using the floor. Let v denote some point to the right of u. If the $\overrightarrow{tv}$ line crosses above u, then the ceiling will block t from v. If the $\overrightarrow{tv}$ line crosses below u, then the floor will block t from v since the floor is blocking t from u. $\square$

Corollary 4.1 *Let $t < u$, t is on the floor (resp. ceiling), u is on the ceiling (resp. floor), and the floor is blocking t from seeing u. It must be that t cannot see any point in $[u, r]$.*

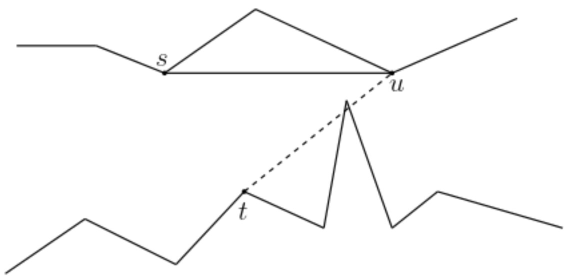

Figure 6: Visualization of Lemma 4.

We obtain an upper bound of 4 by showing that it is impossible to shatter 5 half guards in a monotone polygon. The upper bound proof is obtained by breaking the problem up into different cases. Unfortunately, every viewpoint, when considered by itself without placing any other viewpoints, can be placed when there are 5 guards. However, depending on the location of the guards, certain viewpoint combinations are impossible. We provide a few cases below. Consider a monotone polygon with 5 guards: $\{A, B, C, D, E\}$ such that $A.x \leq B.x \leq C.x \leq D.x \leq E.x$.

Case 1: Let $\{A, C\}$ be on the floor (resp. ceiling) and $\{B, D\}$ be on the opposite side. The position of E does not matter (with respect to the ceiling or floor). We show that it is impossible to place the points $vp(BCE)$ and $vp(ADE)$. Note that $vp(BCE)$ and $vp(ADE)$ must be to the right of, or on the same vertical line, as E.

Case 1a: If $vp(BCE)$ is on the ceiling to the left of $vp(ADE)$, or on same line as $vp(ADE)$, then consider how B must be blocked from $vp(ADE)$. The B guard cannot be blocked from $vp(ADE)$ using the ceiling because of Lemma 2 where $s = B, t = D, u = vp(BCE)$ and $v = vp(ADE)$. The floor must then be used to block B from $vp(ADE)$. By Lemma 3, using $o = A, p = B, q = vp(ADE)$, the A guard would not be able to see $vp(ADE)$. Therefore, B cannot be blocked from $vp(ADE)$. This case is illustrated in Figure 7.

Case 1b: If $vp(ADE)$ is on the ceiling to the left of $vp(BCE)$, or on same line, then consider how C is blocked from seeing $vp(ADE)$. This case is illustrated in Figure 8. Similar to the previous argument, if C is blocked from seeing $vp(ADE)$ using the floor, then by Lemma 4 using $s = A, t = C, u = vp(ADE), v = vp(BCE)$, C cannot see $vp(BCE)$. If the ceiling blocks C from seeing $vp(ADE)$, then by Lemma 3 using $o = A, p = C, q = vp(ADE)$, A is blocked from seeing $vp(ADE)$.

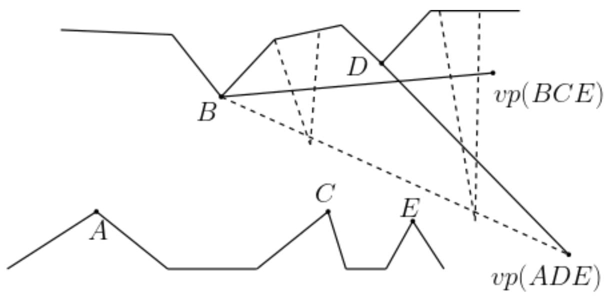

Figure 7: Visualization of Case 1a.

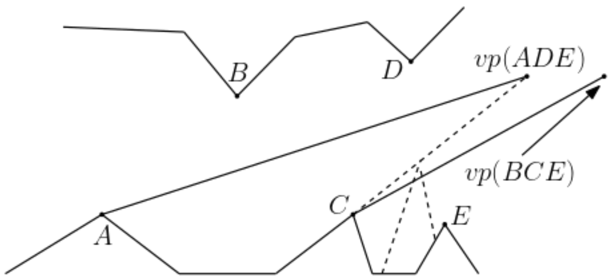

Figure 8: Visualization of Case 1b.

Case 1c: If $vp(BCE)$ is on floor to the left of $vp(ADE)$, or on same line as $vp(ADE)$, then consider how D must be blocked from $vp(BCE)$. If the ceiling blocks D from seeing $vp(BCE)$, then D does not see $vp(ADE)$ by Corollary 4.1 when $t = D, u = vp(BCE), v = vp(ADE)$. If the floor blocks D from seeing $vp(BCE)$, then by Lemma 3 with $o = C, p = D, q = vp(BCE)$, C cannot see $vp(BCE)$.

Case 1d: If $vp(ADE)$ is on floor to the left of $vp(BCE)$, or on same line as $vp(ADE)$, then consider how B is blocked from $vp(ADE)$. If the floor blocks B from $vp(ADE)$, then by Lemma 3 with $o = A, p = B, q = vp(ADE)$, A does not see $vp(ADE)$. If the ceiling blocks B from $vp(ADE)$, then by Corollary 4.1 with $t = B, u = vp(ADE), v = vp(BCE)$, B does not see $vp(BCE)$.

Therefore, $\{A, C\}$ and $\{B, D\}$ cannot be on opposite sides of the polygon. We provide 1 more case.

Case 2: In this case, $\{A, E\}$ are on the floor (resp. ceiling) and $\{B, C, D\}$ are on the opposite side. In this case, it is impossible to place both $vp(BDE)$ and $vp(ACD)$.

Case 2a: The viewpoint $vp(ACD)$ is on the ceiling to the left of $vp(BDE)$ or on same line as $vp(BDE)$. We consider how C is blocked from $vp(BDE)$. We can't block C from $vp(BDE)$ with the ceiling by Lemma 2, where $s = C, t = D, u = vp(ACD), v = vp(BDE)$. If we try to block C from $vp(BDE)$ using the floor, we end up blocking B from $vp(BDE)$ by Lemma 3 with $o = B, p = C, q = vp(BDE)$).

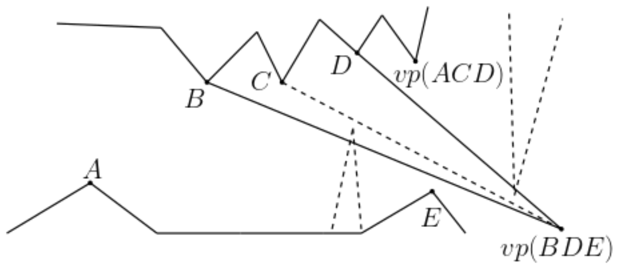

Figure 9: Visualization of Case 2a.

Case 2b: The viewpoint $vp(BDE)$ is on the ceiling to the left of $vp(ACD)$, or on same line as $vp(ACD)$. By Lemma 2 with $s = B, t = C, u = vp(BDE), v = vp(ACD)$, we must use the floor to block B from $vp(ACD)$. However, if we use the floor to block B from $vp(ACD)$, then by Lemma 3 with $o = A, p = B, q = vp(ACD)$, the A guard is blocked from seeing $vp(ACD)$.

Case 2c: The viewpoint $vp(ACD)$ on floor to the left of $vp(BDE)$. In this case, we consider how B is blocked from $vp(ACD)$. If the ceiling blocks B from $vp(ACD)$, then by Corollary 4.1 with $t = B, u = vp(ACD), v = vp(BDE)$, B does not see $vp(BDE)$. If the floor blocks B from $vp(ACD)$, then by Lemma 3 with $o = A, p = B, q = vp(ACD)$, the A guard does not see $vp(ACD)$.

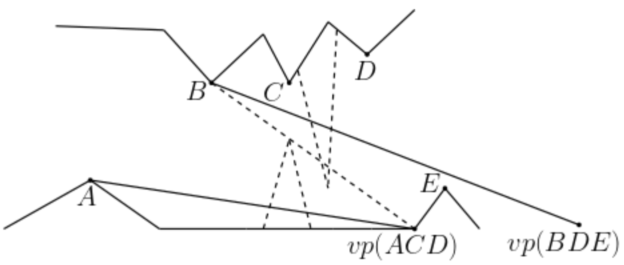

Figure 10: Visualization of Case 2c.

Case 2d: The viewpoint $vp(BDE)$ is on floor to the left of $vp(ACD)$. In this case, consider how C is blocked from seeing $vp(BDE)$. If the floor blocks C from seeing $vp(BDE)$, then by Lemma 3 with $o = B, p = C, q = vp(BDE)$, B would not see $vp(BDE)$. If the ceiling blocks C from $vp(BDE)$, then by Corollary 4.1 with $t = C, u = vp(BDE), v = vp(ACD)$, C would not see $vp(ACD)$.

These cases are just a few examples of how to show the VC dimension of a monotone polygon with half guards is exactly 4. The $2^5 = 32$ cases that we consider are the following: $\{A, C\}$ are on the same side and $\{B, D\}$ are on the opposite side (4 cases), $\{A, E\}$ are on some side and $\{B, C, D\}$ are on the opposite side (2 cases), $\{C, E\}$ are on the same side and $\{A, B, D\}$ are on the opposite side (2 cases), there are any 4 guards that are on the same side (12 cases), $\{A, B\}$ are on the same side and $\{C, D\}$ are on the opposite side (4 cases), $\{A, D\}$ are on the same side and $\{B, C\}$ are on the opposite side (4 cases), $\{B, E\}$ are on the same side and $\{A, C, D\}$ are on the opposite side (2 cases), and $\{A, B, C\}$ are on the same side and $\{D, E\}$ are on the opposite side (2 cases).

These cases give us the following theorem.

Theorem 5 *The VC dimension of half guarding a monotone polygon is exactly 4.*

4 Conclusions

We show the VC dimension exactly for several variants of half guarding in the art gallery problem. The VC dimension for half guarding a terrain is 2 or 3 depending on the assumption of whether or not guards and viewpoints can occupy the same space. The VC dimension for monotone polygons with half guards is exactly 4.

Open problem 6 *What is the VC dimension of half-guarding other variants of the art gallery problem, for example: simple polygons, spiral polygons, orthogonal polygons, etc?*

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