Augmenting plane geometric graphs to meet degree parity constraints

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A fundamental class of problems in graph theory and graph drawing concerns augmenting existing graphs to achieve some desired properties. In this talk, we approach the natural question of augmenting plane geometric graphs to meet degree constraints.

A *geometric graph* $G = (V, E)$ is a graph drawn in the plane such that its vertex set V is a point set in general position (no three points are collinear) and its edge set E is a set of straight-line segments between those points. A geometric graph is *plane* if no two of its edges cross and it is *convex* if its vertices are in convex position. The *visibility graph* of a plane geometric graph G, denoted by $\mathrm{Vis}(G)$, is a geometric graph that has V as its vertex set and two vertices u and v share an edge in $\mathrm{Vis}(G)$ if and only if $uv \notin E$ and uv does not cross any edge in E (so G can be augmented by uv); see Figure 1 (left) for an example.

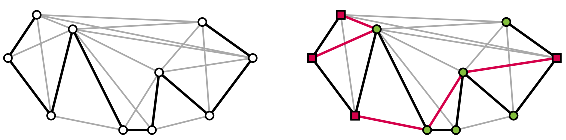

Figure 1: Left: A plane geometric graph G (in black) and its visibility graph $\mathrm{Vis}(G)$ (in gray). Right: A solution set for G and the unhappy vertices (red squares).

Given a plane geometric graph G, the problem we study is to augment it with straight-line edges such that the result is a plane geometric graph in which constraints concerning the degrees of the vertices are met. Even in the simplest version of this problem, where the degree constraints are modulo two, even or odd degree, the problem is NP-hard for general graphs [1], and conjectured to be NP-hard even for trees [1]. In this talk, we answer the question about the tractability for geometric paths; see Table 1.

Table 1: Summary of results on augmenting a plane geometric graph $G = (V, E)$ to meet parity constraints.

Plane geometric G	Any R / $R = V$ / Eulerian
General	NP-hard [1]
Convex	P (Theorem 1)
Paths	P (Theorem 2)
Trees	?

The degree parity constraints can be interpreted as a set of *unhappy* vertices R that would like to change the parity of their degree. We refer to vertices that are not unhappy as *happy*. A subgraph H of $\mathrm{Vis}(G)$ is called a *solution set* for (G, R) if H is crossing-free and the vertices that have odd degree in H are exactly those in R; see Figure 1 (right). Thus, the problem can be reformulated as deciding the existence of a solution set for (G, R). Note that by the handshaking lemma, a solution set can only exist if $|R|$ is even.

For convex plane geometric graph we obtain a surprisingly simple efficient algorithm:

Theorem 1 *Let $G = (V, E)$ be a convex plane geometric graph, and let $R \subseteq V$ be the set of unhappy vertices. There exists a linear-time algorithm to decide whether (G, R) admits a solution set.*

Using this result, we obtain a polynomial-time algorithm for plane geometric paths, solving an open problem by Catana et al. [1].

Theorem 2 *Let $P = (V, E)$ be a plane geometric path and let $R \subseteq V$ with $|R|$ even. There exists an algorithm to decide whether (P, R) admits a solution set in $O(|V| \log |V|)$ time.*

Finally, we characterize those paths that admit a positive answer for any even set R of unhappy vertices.

References

[1] J. C. Catana, A. G. Olaverri, J. Tejel, and J. Urrutia. Plane augmentation of plane graphs to meet parity constraints. *Appl. Math. Comput.*, 386:125513, 2020.

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