Realizations of multiassociahedra via rigidity *

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Abstract

Let $\Delta_k(n)$ denote the simplicial complex of $(k+1)$-crossing-free subsets of edges in $\binom{[n]}{2}$. Here $k, n \in \mathbb{N}$ and $n \geq 2k+1$. Jonsson (2005) proved that (neglecting the short edges that cannot be part of any $(k+1)$-crossing), $\Delta_k(n)$ is a shellable sphere of dimension $k(n-2k-1)-1$, and conjectured it to be polytopal.

Despite considerable effort, the only values of (k, n) for which the conjecture is known to hold are $n \leq 2k+3$ (Pilaud and Santos, 2012) and $(2, 8)$ (Bokowski and Pilaud, 2009). Using ideas from rigidity theory we realize $\Delta_k(n)$ as a polytope for $(k, n) \in \{(2, 9), (2, 10), (3, 10)\}$. We also realize it as a simplicial fan for all $n \leq 13$ and arbitrary k, except the pairs $(3, 12)$ and $(3, 13)$.

1 The multiassociahedron

Triangulations of the convex n-gon P $(n > 2)$ are the facets of an abstract simplicial complex with vertex set $\binom{[n]}{2}$ and defined by taking as simplices all the non-crossing sets of diagonals. This simplicial complex, ignoring the boundary edges $\{i, i+1\}$, is a polytopal sphere of dimension $n-4$ dual to the *associahedron*. (Here and all throughout the paper, indices for vertices of the n-gon are regarded modulo n). A similar complex can be defined if we forbid crossings of more than a certain number k of edges (assuming $n > 2k+1$), instead of forbidding pairwise crossings.

Definition 1 *Two disjoint pairs $\{i, j\}, \{k, l\} \in \binom{[n]}{2}$, with $i < j$ and $k < l$, of $\binom{[n]}{2}$* cross *if $i < k < j < l$ or $k < i < l < j$. That is, if they cross as diagonals of a convex n-gon. A k*-crossing *is a subset of k elements of $\binom{[n]}{2}$ such that every pair cross. A subset of $\binom{[n]}{2}$ is* $(k+1)$-free *if it doesn't contain any $(k+1)$-crossing. A* k-triangulation *is a maximal $(k+1)$-free set. We call $\Delta_k(n)$ the simplicial complex consisting of $(k+1)$-free sets of diagonals, whose facets are the k-triangulations.*

Diagonals of length at most k (with length measured cyclically) cannot participate in any $(k+1)$-crossing. Thus, it makes sense to define the reduced complex $\overline{\Delta}_k(n)$ obtained from $\Delta_k(n)$ by deleting them. We call $\overline{\Delta}_k(n)$ the *multiassociahedron* or k-associahedron.

It was proved in [14, 9] that every k-triangulation of the n-gon has exactly $k(2n-2k-1)$ diagonals. That is, $\Delta_k(n)$ is pure of dimension $k(2n-2k-1)-1$. Jonsson [11] further proved that the reduced version $\overline{\Delta}_k(n)$ is a shellable sphere of dimension $k(n-2k-1)-1$, and conjectured it to be the normal fan of a polytope. See [15, 16, 19] for additional information.

Conjecture 2 ([11]) *$\overline{\Delta}_k(n)$ is a polytopal sphere for every $n \geq 2k+1$; that is, there is a simplicial polytope of dimension $k(n-2k-1)-1$ with $\binom{n}{2} - kn$ vertices whose lattice of proper faces is isomorphic to $\overline{\Delta}_k(n)$.*

Conjecture 2 is easy to prove for $n \leq 2k+3$ [16]. The only additional case for which Jonsson's conjecture is known to hold is $k = 2$ and $n = 8$ [2]. In some additional cases $\overline{\Delta}_k(n)$ has been realized as a complete simplicial fan, but it is open whether this fan is polytopal. This includes the cases $n \leq 2k+4$ [1], $k = 2$ and $n \leq 13$ [13], and $k = 3$ and $n \leq 11$ [1].

Interest in the polytopality of $\overline{\Delta}_k(n)$ also comes from cluster algebras and Coxeter combinatorics. Let $w \in W$ be an element in a Coxeter group W and let Q be a word of a certain length N. Assume that Q contains as a subword a reduced expression for w. The *subword complex* of Q and w is the simplicial complex with vertex set $[N]$ and with faces the subsets of positions that can be deleted from Q and still contain a reduced expression for w. Knutson and Miller [12, Theorem 3.7 and Question 6.4] proved that every subword complex is either a shellable ball or sphere, and they asked whether all spherical subword complexes are polytopal. It was later proved by Stump [19, Theorem 2.1] that $\overline{\Delta}_k(n)$ is a spherical subword complex for the Coxeter system A_{n-2k-1} and, moreover, it is *universal*: every other spherical subword complex of type A appears as a link in some $\overline{\Delta}_k(n)$ [17, Proposition 5.6]. Hence, Conjecture 2 is equivalent to a positive answer in type A to the question of Knutson and Miller.

2 Realizing a simplicial complex as a polytope

If Δ is a pure simplicial complex with vertex set V of dimension $D-1$ (its facets have size D) realizing it as a

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polytope is the same as finding a vector configuration $\mathcal{V} = \{v_i\}_{i\in V} \subset \mathbb{R}^D$ on which Δ yields a *complete simplicial fan*, and then proving the fan to be a *regular triangulation* of $\mathcal{V}$. See [8, Section 9.5] for details.

To prove that an embedding is a simplicial fan we use a version of [8, Corollary 4.5.20] which says that in order for a vector configuration $\mathcal{V} \subset \mathbb{R}^D$ to embed Δ as a simplicial fan the following *Interior Cocircuit Property (ICoP)* is necessary and almost sufficient:

(ICoP) For every facet T of Δ the vectors $\{v_{ij} : \{i,j\} \in T\}$ are independent, and for every two adjacent facets T_1 and T_2 the linear dependence among the vectors $\{v_{ij} : \{i,j\} \in T_1 \cup T_2\}$ has the same sign for the two elements in $T_1 \setminus T_2$ and $T_2 \setminus T_1$.

We apply this to the complex $\overline{\Delta}_k(n)$, for which $V \subset \binom{[n]}{2}$ and $D = k(n-2k-1)$. Each facet is a k-triangulation and two facets are adjacent if and only if the k-triangulations differ by a *flip*, defined as follows:

Proposition 3 (Flips [16, Section 5]) *For every edge f of a k-triangulation T with length greater than k, there is a unique edge $e \in \binom{[n]}{2}$ such that*

$$T \triangle \{e,f\} := T \setminus \{f\} \cup \{e\}$$

is another k-triangulation.

Once we have the complete fan, regularity is equivalent to the feasibility of a system of linear inequalities. We check this with a version of [18, Theorem 3.7], which in turn is related to [8, Proposition 5.2.6(i)].

In some proofs we also use the following fact:

Proposition 4 (Short cycles [5, Cor. 2.9]) *All links of dimension 1 in $\overline{\Delta}_k(n)$ are cycles of length ≤ 5.*

3 Rigidity

Let $\mathbf{p} = (p_1, \dots, p_n)$ be a set of n points in $\mathbb{R}^d$, labelled by $[n]$. Their *bar-and-joint rigidity matrix* is the following $\binom{n}{2} \times nd$ matrix:

$$R(\mathbf{p}) := \begin{pmatrix} p_1 - p_2 & p_2 - p_1 & \dots & 0 \\ p_1 - p_3 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ p_1 - p_n & 0 & \dots & p_n - p_1 \\ 0 & p_2 - p_3 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & p_n - p_{n-1} \end{pmatrix}.$$

The shape of the matrix is as follows: there is a row for each pair $\{i,j\} \in \binom{[n]}{2}$, so rows can be considered labeled by edges in the complete graph K_n. Then, there are n blocks of columns, one for each point p_i and with d columns in each block; in the row of an edge $\{i,j\}$ (or $\{j,i\}$) only the blocks of vertices i and j ae nonzero, and they contain respectively the vectors $p_i - p_j$ and $p_j - p_i$. Put differently, the matrix can be interpreted as a "directed incidence matrix" of the complete graph K_n, except instead of having a single $+1$ and -1 for each edge-vertex incidence we have the d-dimensional vectors $p_i - p_j$ and $p_j - p_i$. For an $E \subset \binom{[n]}{2}$ we denote by $R(\mathbf{p})|_E$ the restriction of $R(\mathbf{p})$ to the rows or elements indexed by E.

Definition 5 *Let $E \subset \binom{[n]}{2}$ be a subset of edges of K_n (equivalently, of rows of $R(\mathbf{p})$). We say that E, or the corresponding subgraph of K_n, is* self-stress-free *or* independent *if the rows of $R(\mathbf{p})|_E$ are linearly independent, and* rigid *or* spanning *if they are linearly spanning (that is, they have the same rank as the whole matrix $R(\mathbf{p})$).*

That is, self-stress-free and rigid graphs are, respectively, the independent and spanning sets in the linear matroid of rows of $R(\mathbf{p})$. We call this matroid the *bar-and-joint rigidity matroid* of $\mathbf{p}$ and denote it $\mathcal{R}(\mathbf{p})$.

The number $k(2n-2k-1) = 2kn - \binom{2k+1}{2}$ of edges in a k-triangulation happens to coincide with the rank of $R(\mathbf{p})$ (or of $\mathcal{R}(\mathbf{p})$) when $\mathbf{p}$ is a set of n points in general position in $\mathbb{R}^{2k}$. This suggest to try to use these matrices to try to embed $\overline{\Delta}_k(n)$ as a simplicial fan. Or, more generally, we can use any of the following two other versions of rigidity, based on matrices of the same shape, size, and rank as $R(\mathbf{p})$, and which fit into the framework of *abstract rigidity matroids of dimension $2k$ on n elements*.

- The *hyperconnectivity* matroid of $\mathbf{p} \subset \mathbb{R}^d$, denoted $\mathcal{H}(\mathbf{p})$, is the matroid of rows of

$$H(\mathbf{p}) := \begin{pmatrix} p_2 & -p_1 & 0 & \dots & 0 & 0 \\ p_3 & 0 & -p_1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ p_n & 0 & 0 & \dots & 0 & -p_1 \\ 0 & p_3 & -p_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & p_n & -p_{n-1} \end{pmatrix}$$

- For points $\mathbf{q} = (q_1, \dots, q_n)$ in $\mathbb{R}^2$ and a parameter $d \in \mathbb{N}$, the d-dimensional *cofactor rigidity* matroid of the points $q_1, \dots, q_n$, which we denote $\mathcal{C}_d(\mathbf{q})$, is the matroid of rows of

$$C_d(\mathbf{q}) := \begin{pmatrix} \mathbf{c}_{12} & -\mathbf{c}_{12} & 0 & \dots & 0 \\ \mathbf{c}_{13} & 0 & -\mathbf{c}_{13} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbf{c}_{1n} & 0 & 0 & \dots & -\mathbf{c}_{1n} \\ 0 & \mathbf{c}_{23} & -\mathbf{c}_{23} & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & -\mathbf{c}_{n-1,n} \end{pmatrix},$$

where the vector $\mathbf{c}_{ij} \in \mathbb{R}^d$ associated to $q_i = (x_i, y_i)$ and $q_j = (x_j, y_j)$ is

$$\mathbf{c}_{ij} = \left((x_i - x_j)^{d-1}, (y_i - y_j)(x_i - x_j)^{d-2}, \ldots, (y_i - y_j)^{d-1}\right).$$

In [3] we prove that these three rigidity theories coincide when the points $\mathbf{p}$ or $\mathbf{q}$ are chosen along the moment curve (for bar-and-joint and hyperconnectivity) and the parabola (for cofactor). More precisely:

Theorem 6 ([3]) *Let $t_1 < \cdots < t_n \in \mathbb{R}$ be real parameters. Let*

$$p_i = (1, t_i, \ldots, t_i^{d-1}) \in \mathbb{R}^d,$$
$$p'_i = (t_i, t_i^2, \ldots, t_i^d) \in \mathbb{R}^d,$$
$$q_i = (t_i, t_i^2) \in \mathbb{R}^2.$$

Then, the matrices $H(p_1, \ldots, p_n)$, $R(p'_1, \ldots, p'_n)$ and $C(q_1, \ldots, q_n)$ can be obtained from one another multiplying on the right by a regular matrix and then multiplying its rows by some positive scalars. In particular, the three matrices define the same oriented matroid.

Definition 7 *We call the matrix $H(p_1, \ldots, p_n)$ in the statement of Theorem 6 the* polynomial d-rigidity matrix with parameters $t_1, \ldots, t_n$. *We denote it $P_d(t_1, \ldots, t_n)$, and denote $\mathcal{P}_d(t_1, \ldots, t_n)$ the corresponding matroid.*

Summing up: for any choice of points $\mathbf{p} \in \mathbb{R}^{2k}$ or $\mathbf{q} \in \mathbb{R}^2$ in general position, the rows of the matrices $R(\mathbf{p})$, $H(\mathbf{p})$ or $C_{2k}(\mathbf{q})$ are a real vector configuration $\mathcal{V} \subset \mathbb{R}^{2kn}$ of rank $k(2n - 2k - 1)$. Moreover, if $\mathbf{p}$ is chosen along the moment curve or $\mathbf{q}$ along the parabola the three theories give linearly equivalent embeddings. The question we address is whether using these vectors as rays we get that the reduced k-associahedron $\overline{\Delta}_k(n)$ is a polytopal fan.

An alternative to realize the fan is "bipartizing" the k-triangulations, as follows:

Definition 8 *The* bipartization *of a graph $G = ([n], E)$ is the graph $G' = ([n] \cup [n]', E')$ where $E' = \{(i, n+1-j) : \{i,j\} \in E, i < j\}$. The* (reduced) bipartization *of a k-triangulation is its bipartization restricted to $[n-k-1] \cup [n-k-1]'$.*

Reduced bipartizations of k-triangulations have $2kn - 3k^2 - 2k$ edges, which is exactly the rank of the hyperconnectivity matroid in dimension k restricted to bipartite graphs. So, we can also use as a vector configuration the rows of $H(\mathbf{p})$ for $\mathbf{p} \subset \mathbb{R}^k$ in general position, restricted or not to the moment curve.

Conjecture 9
1. *k-triangulations of the n-gon are bases in the bar-and-joint rigidity matroid of generic points along the moment curve in dimension $2k$.*
2. *Bipartized k-triangulations of the n-gon are bases in the bar-and-joint rigidity matroid of generic points along the moment curve in dimension k.*

4 Main results

First, as evidence for Conjecture 9 we prove the case $k = 2$:

Theorem 10 ([5, Thm. 1.4]) *2-triangulations are isostatic in dimension 4 for generic positions along the moment curve.*

One may be tempted to change "generic" to "arbitrary" in Conjecture 9, but we show that this stronger conjecture fails in the worst possible way; for every $k \geq 3$ and $n \geq 2k + 3$, the standard positions along the moment curve make some k-triangulation not a basis:

Theorem 11 ([5, Thm. 1.6], [6, Th. 1.13])

1. *The graph $K_9 - \{16, 37, 49\}$ is a 3-triangulation of the n-gon, but it is dependent in the rigidity matroid $\mathcal{C}_6$ for any configuration $\mathbf{q} \subset \mathbb{R}^2$ if the lines through q_1q_6, q_3q_7, and q_4q_9 meet at a point. This occurs, for example, if we take the nine points on the parabola with $t_i = i$.*

2. *The bipartization of the same graph is dependent in $\mathcal{H}_3$ if the cross-ratio between the hyperplanes $(12, 23; 24, 25)$ equals $(2'4', 2'3'; 1'2', 2'5')$, as happens with points along the moment curve with $\mathbf{t} = (1, 3, 4, 5, 7, 1, 3, 4, 5, 7)$.*

In fact, for $n \leq 2k + 3$ we can characterize exactly what positions realize $\overline{\Delta}_k(n)$ as a fan, for cofactor rigidity (and, in particular, for the other two forms of rigidity with positions along the moment curve), and for bipartite rigidity along the moment curve. In the case $n = 2k + 3$ this is governed by the geometry of the star-polygon formed by the k-relevant edges. More precisely, we call "big side" of each relevant edge (that is, edge of $k + 1$) in a $(2k + 3)$-gon the open half-plane containing $k + 1$ vertices:

Theorem 12 ([5, Thm. 3.14], [6, Thm. 5.6])

1. *For $n = 2k + 2$, any choice of $q_1, \ldots, q_{2k+2} \in \mathbb{R}^2$ in convex position for cofactor rigidity, and any choice of $t_1 < \cdots < t_{k+1}, t'_1 < \cdots < t'_{k+1}$ in the moment curve for bipartite rigidity, realizes $\overline{\Delta}_k(2k + 2)$ as a polytopal fan.*

2. *Let $q_1, q_2, \ldots, q_{2k+3} \in \mathbb{R}^2$ be in convex position. $\overline{\Delta}_k(2k + 3)$ is realized by $C_{2k}(\mathbf{q})$ as a complete fan if and only if the big sides of all relevant edges have a non-empty intersection.*

3. *Let $t_1 < \cdots < t_{k+2}, t'_1 < \cdots < t'_{k+2}$ be parameters for the vertices of $K_{k+2,k+2}$ in the moment curve. $\overline{\Delta}_k(2k+3)$ is realized by $P_k(\mathbf{t})$ as a complete fan if and only if one of the following holds:*
   - $k = 2$.
   - *$k = 3$ and the cross-ratio $(1, 3; 4, 5)$ is greater than $(4', 3'; 1', 5')$.*
   - *$k \geq 4$ and the cross-ratio $(i_1, i_2; i_3, k+2)$ is greater than $((k+3-i_1)', (k+2-i_2)'; (k+3-i_3)', (k+2)')$, for any i_1, i_2, i_3 with $2 \leq i_1 < i_2 < i_3 - 1 \leq k$.*

   Here, by cross-ratio between four points, we mean the cross-ratio between their parameters t.

Interestingly, from parts (2) and (3) of this result it is quite easy to show that *no positions* of points along the moment curve realize $\overline{\Delta}_k(n)$, for several values of k and n:

Corollary 13 ([5, Thm. 1.7], [6, Thm. 1.14])

1. *If $k \geq 3, n \geq 2k+6$ then no choice of points $\mathbf{q} \subset \mathbb{R}^2$ in convex position realizes $\overline{\Delta}_k(n)$ as a fan via cofactor rigidity.*
2. *If $k = 3, n \geq 12$, or $k \geq 4, n \geq 2k+4$, then no choice of points $\mathbf{t} \in \mathbb{R}^{2(n-k-1)}$ in the moment curve realizes $\overline{\Delta}_k(n)$ as a fan via cofactor rigidity.*

Observe that this is not a counter-example to Conjecture 9, which is only about linear independence of the vectors generating each facet of the fan, not about the fan itself.

Finally, for every $n \leq 13$ we have experimentally found positions along the moment curve realizing $\overline{\Delta}_k(n)$ as a fan, except in the cases $(n, k) \in \{(3, 12), (3, 13)\}$ which are forbidden by Corollary 13. For many of them we have also realized the polytope:

Theorem 14 ([5, Lem. 4.13 & 4.14], [6, Thm. 5.10]) *Let $\mathbf{t} = \{1, 2, \ldots, n\}$ be standard positions for the parameters. Then:*

1. *Standard positions realize $\overline{\Delta}_2(n)$ as the normal fan of a polytope for $P_4(\mathbf{t})$ with the original graph if $n \leq 9$, and for $P_2(\mathbf{t})$ with the bipartized graph if $n \leq 8$.*
2. *The non-standard positions $\mathbf{t} = (-2, 1, 2, 3, 4, 5, 6, 7, 9, 20)$ for $P_4(\mathbf{t})$ with the original graph, and the near-lexicographic positions $t_i = t'_i = 2^{(i-1)^2}$ for $P_2(\mathbf{t})$ with the bipartized graph, realize $\overline{\Delta}_2(10)$ as the normal fan of a polytope.*
3. *Standard positions realize $\overline{\Delta}_2(n)$ as a complete fan for all $n \leq 13$ with both forms of rigidity.*
4. *Equispaced positions along the circle with the original graph realize $\overline{\Delta}_k(n)$ as a fan for $(k, n) \in \{(3, 10), (3, 11), (4, 12), (4, 13)\}$. The first one is polytopal.*
5. *The positions $\mathbf{t} = (0, 1, 31, 32, 42, 67, 100)$ at both sides with bipartite rigidity realize $\overline{\Delta}_3(11)$ as a fan.*

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