On the rectilinear crossing number of complete balanced multipartite graphs and layered graphs

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Abstract

A rectilinear drawing of a graph is a drawing of the graph in the plane in which the edges are drawn as straight-line segments. The rectilinear crossing number of a graph is the minimum number of pairs of edges that cross over all rectilinear drawings of the graph. Let $n \geq r$ be positive integers. The graph K_n^r, is the complete r-partite graph on n vertices, in which every set of the partition has at least $\lfloor n/r \rfloor$ vertices. The layered graph, L_n^r, is an r-partite graph on n vertices, in which for every $1 \leq i \leq r-1$, all vertices in the i-th partition are adjacent to all vertices in the $(i+1)$-th partition. In this paper, we give upper bounds on the rectilinear crossing numbers of K_n^r and L_n^r.

1 Introduction

Let G be a graph on n vertices and let D be a drawing of G. The crossing number of D is the number, $\text{cr}(D)$, of pairs of edges that cross in D. The *crossing number* of G is the minimum crossing number, $\text{cr}(G)$, over all drawings of G in the plane. A *rectilinear drawing* of G is a drawing of G in the plane in which its vertices are points in general position, and its edges are drawn as straight-line segments joining these points. The *rectilinear crossing number* of G, is the minimum crossing number, $\overline{\text{cr}}(G)$, over all rectilinear drawings of G in the plane.

Computing crossing and rectilinear crossing numbers of graphs are important problems in Graph Theory and Combinatorial Geometry. For a comprehensive review of the literature on crossing numbers, we refer the reader to Schaefer's book [1].

Most of the research on crossing numbers have been focused around the complete graph, K_n, and the complete bipartite graph $K_{n,m}$. For the complete graph, Hill [2] gave the following drawing of K_n; see Figure 1 for an example. Place half of the vertices equidistantly on the top circle of a cylinder, and the other half equidistantly on the bottom circle. Join the vertices with geodesics on the cylinder. Hill showed that the following number, $H(n)$, is the crossing number of this drawing, and it is now conjectured to be optimal. Let

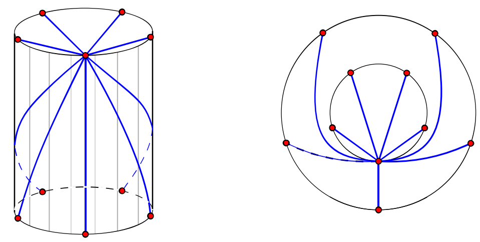

Figure 1: An example of Hill's drawings of K_8, where here for convenience only the edges of one vertex are drawn. Left: The drawing on a cylinder. Right: An equivalent representation of Hill's drawings via concentric cycles.

$$H(n) := \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

Conjecture 1 (Harary-Hill [3])

$$\text{cr}(K_n) = H(n).$$

For the complete bipartite graph, Zarankiewicz gave a rectilinear drawing with the following number, $Z(n,m)$, as crossing number of this drawing, and it is now conjectured to be optimal. Let

$$Z(n,m) := \left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor$$

and

$$Z(n) := Z(n,n).$$

Conjecture 2 (Zarankiewicz [4])

$$\text{cr}(K_{n,m}) = Z(n,m).$$

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The number $Z(n,m)$ is also conjectured to be the general optimal crossing number, directly implying the following conjecture.

Conjecture 3

$$\overline{\text{cr}}(K_{n,m}) = \text{cr}(K_{n,m}).$$

Much less is known for the rectilinear crossing number of the complete graph. For $n \geq 10$, it is known that

$$\text{cr}(K_n) < \overline{\text{cr}}(K_n).$$

In contrast to the case of the complete bipartite graph, there is no conjectured value for $\overline{\text{cr}}(K_n)$, nor drawings conjectured to be optimal. The best bounds to date are

$$0.379972\binom{n}{4} < \overline{\text{cr}}(K_n) < 0.380445\binom{n}{4} + O(n^3).$$

The lower bound is due to Ábrego, Fernández-Merchant, Leaños, and Salazar [5], and the upper bound to Aichholzer, Duque, Fabila-Monroy, García-Quintero, and Hidalgo-Toscano [6]. It is known that

$$\lim_{n\to\infty} \frac{\overline{\text{cr}}(K_n)}{\binom{n}{4}} = \overline{q},$$

for some positive constant $\overline{q}$; this constant is known as the *rectilinear crossing constant*.

Let $K_{n_1 n_2 \ldots n_r}$ be the complete r-partite graph with n_i vertices in the i-th set of the partition; and let K_n^r be the complete balanced r-partite graph in which there are at least $\lfloor n/r \rfloor$ vertices in every partition set. Harborth [7] gave a drawing that provides an upper bound for $\text{cr}(K_{n_1 n_2 \ldots n_r})$; and gave an explicit formula for this number. He claims that for the case of $r = 3$, his drawing can be made rectilinear. More recently, Gethner, Hogben, Lidický, Pfender, Ruiz and Young [8] independently studied the problem of the crossing number and rectilinear crossing numbers of complete balanced r-partite graphs. For $r = 3$, they obtain the same bound as Harborth; and their drawing is rectilinear.

Let $H(n,r)$ be the number of crossings in Harborth's drawing for $\text{cr}(K_n^r)$. Due to the complexity of the formula, we use the following approximation to $H(n,r)$ instead. (All missing proofs can be found in the Appendix)

Lemma 4 *If n is a multiple of r, then*

$$H(n,r) \leq \frac{1}{16}\left(\frac{r-1}{r}\right)^2 \left(\frac{n^4}{4} - 2n^3\right) + O(n^2).$$

In this paper, we mainly focus on the rectilinear crossing number of K_n^r. If n is fixed and r tends to n, then K_n^r tends to K_n. Thus, we believe that studying the rectilinear crossing number of K_n^r might shed some light on how optimal rectilinear drawings of K_n look like.

2 Random Embeddings into Drawings of K_n with Few Crossings

Suppose that we have a drawing (that can be rectilinear but doesn't have to be) D' of K_n. If $\text{cr}(D')$ is small, it might be a good idea to use this drawing to produce a drawing of a graph G on n vertices. Let D be the drawing of G that is produced by mapping the vertices of G randomly to the vertices of D', and where the edges are drawn as their corresponding edges of D'. We call D a *random embedding* of G into D'. In every 4-tuple of vertices of D', there are three pairs of independent edges, which could cross. Of these three pairs at most one pair is crossing. For every pair of independent edges of G, we have a possible crossing in D; thus, the probability that this pair of edges is mapped to a pair of crossing edges is equal to

$$\frac{1}{3} \cdot \frac{\text{cr}(D')}{\binom{n}{4}}.$$

By defining, for every pair of independent edges of G, an indicator random variable with value equal to one if the edges cross and zero otherwise, we obtain the following result where $||G||$ is the number of edges in G and $d(v)$ is the degree of a vertex v.

Lemma 5

$$\text{E}(\text{cr}(D)) = \frac{cr(D')}{3\binom{n}{4}} \left(\binom{||G||}{2} - \sum_{v \in V(G)} \binom{d(v)}{2} \right).$$

Suppose that n is a multiple of r. For the crossing number of K_n^r, we use Lemma 5 and Hill's drawing of K_n to obtain the following.

Theorem 6 *Suppose that n is a multiple of r. Let D be a random embedding of K_n^r into Hill's drawing of K_n. Then,*

$$E(\text{cr}(D)) \leq \frac{1}{16}\left(\frac{r-1}{r}\right)^2 \left(\frac{n^4}{4} - \frac{3n^3}{2}\right) + O(n^2).$$

In [8], the authors obtain same bound on $\text{cr}(K_n^r)$ by considering a random mapping of the vertices of K_n^r into a sphere, and then joining the corresponding vertices with geodesics. This type of drawing is called a *random geodesic spherical drawing.* In 1965, Moon [9], showed that the expected number of crossings of a random geodesic spherical drawing of K_n is equal to

$$\frac{1}{16}\binom{n}{2}\binom{n-2}{2} = H(n) - O(n^3);$$

which explains why the bound of Theorem 6 matches the bound of [8].

Note that by Lemma 4 together with Theorem 6, it holds that

$$E(\text{cr}(D)) - H(n,r) \leq \frac{1}{32}\left(\frac{r-1}{r}\right)^2 n^3 + O(n^2) = O(n^3).$$

Thus, the random embedding gives an upper bound on cr(D) that matches the conjectured value up to the leading term, but it is a little worse in the lower terms.

2.1 Rectilinear crossing number of K_n^r

Let $\overline{D}$ be a random embedding of K_n^r into an optimal rectilinear drawing of K_n.

Theorem 7 *Let r be a positive integer and n a multiple of r. Then*

$$\begin{aligned}\overline{\mathrm{cr}}(K_n^r) &\leq E(\mathrm{cr}(\overline{D}))\\ &\leq \frac{\overline{q}}{4!}\left(\frac{r-1}{r}\right)^2 n^4 + o(n^4)\\ &< 0.015852\left(\frac{r-1}{r}\right)^2 n^4 + o(n^4).\end{aligned}$$

For a lower bound we have the following.

Theorem 8 *Let r be a positive integer and n a multiple of r. Then*

$$\overline{\mathrm{cr}}(K_n^r) \geq \overline{\mathrm{cr}}(K_r)\left(\frac{n}{r}\right)^4.$$

Theorems 7 and 8 imply the following.

Corollary 9 *Let $r = r(n)$ be a monotone function of n such that $r \to \infty$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} \frac{\overline{\mathrm{cr}}(K_n^r)}{\binom{n}{4}} = \overline{q}.$$

In both [7] and [8], it is conjectured that

$$\mathrm{cr}\left(K_n^3\right) = \overline{\mathrm{cr}}\left(K_n^3\right).$$

Using the order type database [10], we have verified the following.

Observation 10

$$\overline{\mathrm{cr}}\left(K_8^4\right) = 8 \textit{ and } \overline{\mathrm{cr}}\left(K_9^4\right) = 15.$$

On the other hand

$$\mathrm{cr}\left(K_8^4\right) \leq H(8,4) = 6 \text{ and } \mathrm{cr}\left(K_9^4\right) \leq H(9,4) = 15.$$

See Figure 2 for an example. From the above results we conjecture the following.

Conjecture 11 *There exists a natural number $n_0 > 9$ such that for all $n \geq n_0$,*

$$\mathrm{cr}\left(K_n^4\right) < \overline{\mathrm{cr}}\left(K_n^4\right).$$

Further, we pose the following problem.

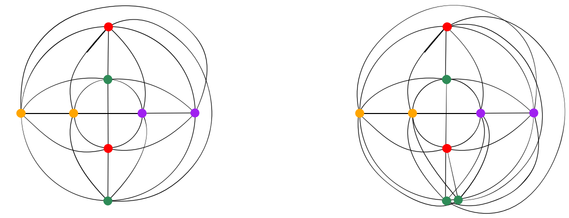

Figure 2: A drawing of K_8^4 with 6 crossings (left) and K_9^4 with 15 crossings (right).

Open problem 12 *Let $r \geq 4$ be a constant positive integer. Does*

$$\lim_{n\to\infty} \frac{\overline{\mathrm{cr}}(K_n^r)}{\binom{n}{4}} = \overline{q}\left(\frac{r-1}{r}\right)^2 ?$$

We believe that finding a good rectilinear drawing of K_n^r, even for the case of $r = 4$, will help in understanding how crossing optimal rectilinear drawings of K_n look like. Theorem 7 implies that

$$\overline{\mathrm{cr}}(K_n^4) \leq 0.00892n^4 + O(n^3).$$

We have found an explicit rectilinear drawing of K_n^4 with

$$0.00953n^4 + O(n^3)$$

crossings.

2.2 Layered graphs

Let r be a positive integer and let n be a multiple of r. The *layered graph*, L_n^r, is the graph defined as follows. Its vertex set is partitioned into sets $V_1, \ldots, V_r$, each consisting of n/r vertices. We call the set V_i, the i-th layer of L_n^r. The edge set of L_n^r is given by

$$\{uv : u \in L_i \text{ and } v \in L_{i+1}, \text{ for } i = 1, \ldots, r-1\};$$

that is, the edges are exactly all possible edges between vertices on consecutive layers. The random embedding into crossing optimal drawings of K_n, seems to give drawings of K_n^r with crossings close to the minimum value. In this section, we show that for layered graphs, the random embedding gives drawings with considerably more crossings than the optimal drawings. Note that $L_n^2 = K_{n/2,n/2}$, and $L_n^3 = K_{2n/3,n/3}$. In the reminder of the section assume that $r \geq 4$, and for convenience assume also that n/r is even. Using the random embedding technique into Hill's drawing of K_n, we obtain the following upper bound.

Theorem 13

$$\mathrm{cr}(L_n^r) \leq \frac{(r-1)^2}{16r^4} n^4 + O(n^3).$$

We can improve on this bound with the following rectilinear drawing, $\overline{D}$, of L_n^r. Since we are describing a rectilinear drawing it is sufficient to specify the location of the vertices of $\overline{D}$:

- $$V_1 := \left\{ \left(1 + \frac{1}{2^j}, 0\right) : 1 \le j \le \frac{n}{r} \right\};$$
- for $2 \le i \le r-1$,
  $$V_i := \left\{ (i,j) : 1 \le j \le \frac{n}{2r} \right\} \cup \left\{ (i,-j) : 1 \le j \le \frac{n}{2r} \right\};$$
  and
- $$V_r := \left\{ \left(r + \frac{1}{2^j}, 0\right) : 1 \le j \le \frac{n}{r} \right\}.$$

Thus, the first and r-th layer are horizontal and the remaining layers are vertical. The vertices of $\overline{D}$ are not in general position; a small random perturbation of the vertices of $\overline{D}$ is sufficient to ensure general position, while at the same time not changing the number of crossings.

Theorem 14

$$\operatorname{cr}(\overline{D}) = \frac{r-2}{4r^4} n^4 + O(n^3).$$

The leading constant in Theorem 14 is better, for all $r \ge 4$, than the constant obtained by the random embedding into Hill's drawing of K_n.

For $r = 4$, Theorem 14 implies that

$$\operatorname{cr}(\overline{D}) \le \frac{3}{1024} n^4 + O(n^3) \le 0.00293 n^4 + O(n^3).$$

We have found a rectilinear drawing of L_n^4 with considerable fewer crossings; see Figure 3 for a depiction.

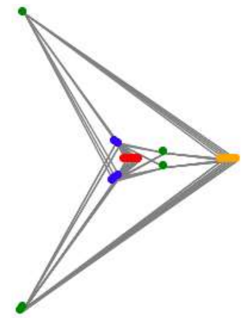

Figure 3: A rectilinear drawing of L_{20}^4

Proposition 15 *There exists a rectilinear drawing D' of L_n^4 such that*

$$\operatorname{cr}(D') = \frac{56}{32768} n^4 + O(n^3) \le 0.00171 n^4 + O(n^3).$$

We believe that the bound of Theorem 14 is far from optimal, and that computing $\operatorname{cr}(L_n^r)$ and $\overline{\operatorname{cr}}(L_n^r)$ are interesting problems.

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