A fitting problem in three dimension

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Abstract

Let P be a set of n points in $\mathbb{R}^3$ in general position, and let $RCH(P)$ be the rectilinear convex hull of P. In this paper we use an efficient $O(n \log^2 n)$ time and $O(n \log n)$ space algorithm to compute and maintain the set of vertices of the rectilinear convex hull of P as we rotate $\mathbb{R}^3$ around the Z-axis to obtain an improvement of the time complexity in an optimization algorithm for a fitting problem in $\mathbb{R}^3$.

1 Introduction

Let P be a set of n points in $\mathbb{R}^3$ in general position, and let $RCH(P)$ be the rectilinear convex hull of P. An *open octant* in $\mathbb{R}^3$ is the intersection of the three open halfspaces, whose supporting planes are perpendicular to the X-axis, to the Y-axis, and to the Z-axis, respectively. An octant is called P-free if it contains no elements of P. The rectilinear convex hull of a set of points in $\mathbb{R}^3$ is defined as

$$RCH(P) = \mathbb{R}^3 \setminus \bigcup_{W \in \mathcal{W}(P)} W,$$

where $\mathcal{W}(P)$ is the set of P-free open octants of $\mathbb{R}^3$.

Theorem 1 *[2] The rectilinear convex hull of P, $RCH(P)$, can be computed in optimal $O(n \log n)$ time and $O(n)$ space.*

If we do rotations of the X- and Y-axis around the Z-axis by an angle θ in the clockwise direction, instead of octants we get θ-octants, and the corresponding rectilinear convex hulls generated $RCH_\theta(P)$. Thus, an open θ-octant is the intersection of three open halfspaces whose supporting planes are orthogonal to three mutually orthogonal lines through the origin X_θ, Y_θ, and Z. An open θ-octant is called P-free if it contains no elements of P. The θ-rectilinear convex hull $RCH_\theta(P)$ of a point set P is defined as

$$RCH_\theta(P) \ = \ \mathbb{R}^3 \setminus \bigcup_{W \in \mathcal{W}_\theta(P)} W,$$

where $\mathcal{W}_\theta(P)$ denotes the set of all P-free *open θ-octants*. The points of P are labeled $\{p_1, \ldots, p_n\}$ from top to bottom by decreasing z-coordinates.

Theorem 2 *[2] Maintaining the elements of P that belong to the boundary of $RCH_\theta(P)$ as $\theta \in [0, 2\pi]$ can be done in $O(n \log^2 n)$ time and $O(n \log n)$ space. The algorithm stores the set of angular intervals in $[0, \pi]$ at which the points are θ-active.*

2 A 2-fitting problem in 3D

The *oriented 2-fitting problem* [1] is defined as follows: Given a point set P in $\mathbb{R}^3$, find a plane Π, called the *splitting plane* of P (assume that Π is parallel to the XY-plane), and four parallel halfplanes $\pi_1, \pi_2, \pi_3, \pi_4$, called the *supporting halfplanes* of P, such that:

1. Π splits P into two non-empty subsets P_1 and P_2, i.e., $\{P_1, P_2\}$ is the bipartition of P produced by the splitting-plane Π.

2. $\pi_1, \pi_2, \pi_3, \pi_4$ are orthogonal to Π, π_1 and π_2 lie above Π, and π_3 and π_4 lie below Π, each one of $\pi_1, \pi_2, \pi_3, \pi_4$ containing at least a point of P. The point sets P_1 and P_2 are contained between π_1 and π_2, and π_3 and π_4, respectively. See Figure 1.

3. The maximum of ϵ_1 and ϵ_2 is minimized, where ϵ_1 is the error tolerance of P_1 with respect to π_1 and π_2, and ϵ_2 is the error tolerance of P_2 with respect to π_3 and π_4.

4. The solution for the 2-fitting problem is given by the mid halfplane of the supporting halfplanes π_1 and π_2 and the mid halfplane of the supporting halfplanes π_3 and π_4.

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The error tolerances ϵ_1 and ϵ_2 are defined by the Euclidean distances between the two parallel supporting-halfplanes on either sides of Π. The problem consists in getting the bipartition $\{P_1, P_2\}$ of P such that $\max\{\epsilon_1, \epsilon_2\}$ is minimum. It is a min-max problem. See Figure 1. If the orientation of the splitting plane

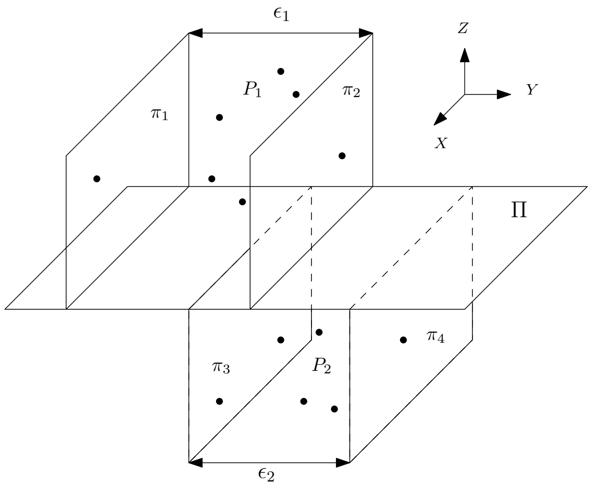

Figure 1: The splitting-plan plane Π and the two pairs of parallel supporting-halfplanes.

is fixed, the problem can be solved in $O(n^2)$ time and $O(n)$ space, as proved by Díaz-Báñez et al. [1]. We will design an algorithm that bounds the solution for this case: when the orientation of the splitting plane is fixed. The complexities are smaller than those in the algorithm Díaz-Báñez et al. [1]. In our algorithm, instead of doing a sequence of bipartitions of P, we will maintaining $RCH_\theta(P)$ as we rotate the space around the Z-axis, and compute optimal solutions in each of the linear number of events at which the $RCH_\theta(P)$ changes for $\theta \in [0, \pi]$.

We assume that the splitting-plane Π is parallel to the XY plane, all the points of P are above the XY plane and sorted with respect to the z-coordinate, where p_1 is the point with the largest z-coordinate, and p_n is the point with the smallest z-coordinate. We also assume that the Z-axis passes through p_1, and thus, its coordinates do not change as we rotate $\mathbb{R}^3$ around the Z-axis, and p_1 and p_n are always in the boundary of $RCH_\theta(P)$.

Suppose that the supporting halfplanes have normal unit vectors. Thus, the 2-fitting problem reduces to computing four parallel supporting halfplanes $\pi_1, \pi_2, \pi_3, \pi_4$ of a bipartition $\{P_1, P_2\}$ of P, and therefore, to computing four points in the boundary of $RCH_\theta(P)$, for some $\theta \in [0, \pi]$. See Figure 2.

We will discretize the problem by considering the angular sub-intervals of $[0, \pi]$ such that in each sub-interval the points in the boundary of $RCH_\theta(P)$ do not change. By Theorem 2, their number is linear in n. Then, we will show how to optimize the error tolerance at the endpoints of each of these sub-intervals.

Recall that a point $p \in P$ is said to be θ-active if at least one of the p^θ-octants is P-free. The definition of a θ-active point considering an octant can be easily adapted to considering a *dihedral* (two perpendicular and axis-parallel planes) as follows:

Definition 3 *Let $p \in P$ and $\{s, t\} \in \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}$. We say that p is a $\theta_{\{s,t\}}$-active point if p is θ-active for both the s-th and t-th octants.*

For example, a point $p \in P$ is $\theta_{\{1,2\}}$-active if p is θ-active for both the first and second octants. In fact, the union of the first and second p^θ-octants is a dihedral, which is P-free and its edge goes through p.

Lemma 4 *The boundary of the projection of $RCH_\theta(P)$ on the ZY_θ plane is formed by the points of P which are $\theta_{\{1,2\}}$-active, $\theta_{\{3,4\}}$-active, $\theta_{\{5,6\}}$-active, and $\theta_{\{7,8\}}$-active in the direction defined by the ZY_θ plane in the unit circle S^1. Thus, the four staircases of the boundary of the projection of $RCH_\theta(P)$ on the ZY_θ plane are as follows: the first staircase is formed by the $\theta_{\{1,2\}}$-active points, the second staircase is formed by the $\theta_{\{3,4\}}$-active points, the third staircase is formed by the $\theta_{\{5,6\}}$-active points, and the fourth staircase is formed by the $\theta_{\{7,8\}}$-active points.*

Proof. The proof follows by observing that any point p in the interior of the projection of $RCH_\theta(P)$ on the ZY_θ plane is dominated by at least one point for each of the four quadrants, and it is so because p is not $\theta_{\{s,t\}}$-active for any $\{s, t\} \in \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}$, see Figure 2. Furthermore, the rightmost point in the projection on the ZY_θ is both $\theta_{\{1,2\}}$-active and $\theta_{\{7,8\}}$-active, and the leftmost point is both $\theta_{\{3,4\}}$-active and $\theta_{\{5,6\}}$-active. □

Let p^θ_{left} and p^θ_{right} denote, respectively, the leftmost and rightmost points of the projection of P on the plane ZY_θ. For any angle θ, let L_θ be the list consisting of the $\theta_{\{1,2\}}$-active points and the $\theta_{\{5,6\}}$-active points of P, sorted in decreasing order of their z-coordinate. By simplicity, we assume with loss of generality that no two elements of P have the same z-coordinate. Let $m = O(n)$ denote the number of elements of L_θ and let $z_1, z_2, \ldots, z_m$ denote the sorted elements of L_θ. The $\theta_{\{1,2\}}$-active points of L_θ, those of the first staircase of $RCH_\theta(P)$ are colored red, and the $\theta_{\{5,6\}}$-active points, those of the third staircase of $RCH_\theta(P)$, are colored blue (see the red and blue staircases of Figure 2, containing as vertices the red and blue elements of L_θ, respectively). We can represent L_θ as a standard binary search tree, with extra $O(1)$-size data at each node, such that inserting/deleting an element can be done in $O(\log n)$ time, and also the following queries can all be answered in $O(\log n)$ time:

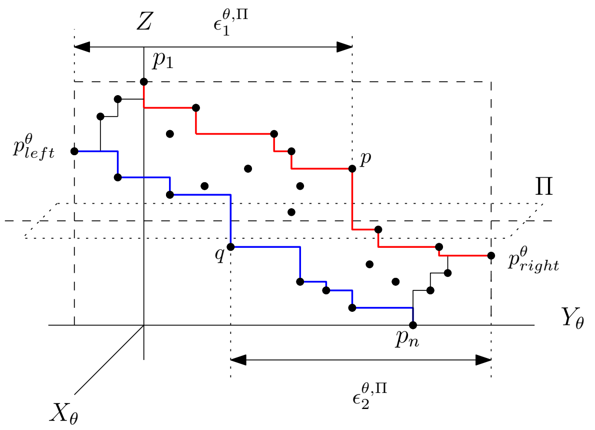

Figure 2: Projection of the $RCH_\theta(P)$ on the ZY_θ plane. The bipartition plane Π is determined by the $\theta_{\{1,2\}}$-active point p and the $\theta_{\{5,6\}}$-active point q, which determine the error tolerance functions $\epsilon_1^{\theta,\Pi}$ and $\epsilon_2^{\theta,\Pi}$.

(1) Given an element in the list, retrieve its position.

(2) Given a position $j \in \{1, 2, \ldots, m\}$, retrieve the rightmost *red* element in the sublist $z_1, z_2, \ldots, z_j$.

(3) Given a position $j \in \{1, 2, \ldots, m\}$, retrieve the leftmost *blue* element in the sublist $z_j, z_{j+1}, \ldots, z_m$.

By simplicity in the explanation, we will assume that p_{left}^{θ} is always above p_{right}^{θ} in the z-coordinate order. Hence, any bipartition plane Π must have p_{left}^{θ} above it and p_{right}^{θ} below it. Furthermore, Π is determined by the closest $\theta_{\{1,2\}}$-active point p above Π and the closest $\theta_{\{5,6\}}$-active point q below Π (see Figure 2). This is why the list L_θ is for the $\theta_{\{1,2\}}$-active and $\theta_{\{5,6\}}$-active points. If the assumption is not considered, then our arguments must include a similar list with the $\theta_{\{3,4\}}$-active and $\theta_{\{7,8\}}$-active points for the situations in which p_{left}^{θ} is below p_{right}^{θ}.

The next facts are the keys for the algorithm:

1. Since each point of P can change its condition of being $\theta_{\{s,t\}}$-active a constant number of times, then the total number of times there is a change in some of the four staircases, hence in L_θ, is $O(n)$. Thus, we have $O(n)$ intervals of $[0, \pi]$ with no change in the staircases. We can then define the sequence Θ of the $N = O(n)$ angles $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_N = \pi$, such that for each interval $[\theta_i, \theta_{i+1})$, $i = 0, 1, \ldots, N-1$ the list L_θ do not change.

2. For an angle $\theta \in [0, \pi]$ and a point p of P, let p^θ be the projection of p on ZY_θ, and let α_p be the angle formed by the X-axis and the line through the origin O and the projection of p on the XY-plane. For any point q, let $d(q, Z)$ denote the distance from q to the Z-axis. We have that $d(p^\theta, Z) = d(p, Z) \cdot \cos(\alpha_p - \theta)$, which is a function depending only on θ since $d(p, Z)$ and α_p are constants.

3. For a fixed angle $\theta \in [0, \pi]$, a bipartition of P by a plane Π induces a partition of the list $L_\theta = z_1, z_2, \ldots, z_m$ into two sublists: $z_1, z_2, \ldots, z_k$ with the elements above Π, and $z_{k+1}, z_{k+2}, \ldots, z_m$ with the elements below Π. And vice versa, every such a partition of L_θ into two lists induces a plane Π that bipartitions P. Let the $\theta_{\{1,2\}}$-active point p and the $\theta_{\{5,6\}}$-active point q be the *witnesses* of this bipartition. That is, p is the rightmost red element in $z_1, z_2, \ldots, z_k$, and q is the leftmost blue element in $z_{k+1}, z_{k+2}, \ldots, z_m$ (see Figure 2). The error tolerances for this bipartition, denoted $\epsilon_1^{\theta,\Pi}$ and $\epsilon_2^{\theta,\Pi}$, are given by the distances

$$\epsilon_1^{\theta,\Pi} = d(p_{left}^{\theta}, Z) + d(p^\theta, Z) \quad \text{and}$$

$$\epsilon_2^{\theta,\Pi} = d(p_{right}^{\theta}, Z) \pm d(q^\theta, Z),$$

where the $+$ or $-$ depends on whether q^θ is to the left or right of the Z-axis in the ZY_θ plane. Note that when moving Π upwards, the functions $\epsilon_1^{\theta,\Pi}$ and $\epsilon_2^{\theta,\Pi}$ are non-increasing and non-decreasing, respectively. Hence, to find an optimal Π for a given angle θ, we can perform a binary search in the range $\{k_1, k_1+1, \ldots, k_2-1\} \subset \{1, 2, \ldots, m-1\}$ to find an optimal partition $z_1, z_2, \ldots, z_k$ and $z_{k+1}, \ldots, z_m$ of L_θ, where k_1 and k_2 are the positions of p_{left}^{θ} and p_{right}^{θ} in L_θ, respectively.

   The binary search does the following steps for a given value $k \in \{k_1, k_1+1, \ldots, k_2-1\}$: Consider a bipartition plane Π induced by the partition $z_1, z_2, \ldots, z_k$ and $z_{k+1}, \ldots, z_m$ of L_θ, and find the witnesses points p and q, each in $O(\log n)$ time by using the queries of the tree supporting L_θ. Then, compute $\epsilon_1^{\theta,\Pi}$ and $\epsilon_2^{\theta,\Pi}$ in constant time. If $\epsilon_1^{\theta,\Pi} = \epsilon_2^{\theta,\Pi}$, then stop the search. Otherwise, if $\epsilon_1^{\theta,\Pi} < \epsilon_2^{\theta,\Pi}$ (resp. $\epsilon_1^{\theta,\Pi} > \epsilon_2^{\theta,\Pi}$), then we increase (resp. decrease) the value of k accordingly with the binary search and repeat. We return the value of k visited by the search that minimizes $\max\{\epsilon_1^{\theta,\Pi}, \epsilon_2^{\theta,\Pi}\}$. This search makes $O(\log n)$ steps, each in $O(\log n)$ time, thus it costs $O(\log^2 n)$ time.

4. Let θ_i and θ_{i+1} be two consecutive angles of the sequence Θ. It may happen for some angle $\theta \in (\theta_i, \theta_{i+1})$, and some bipartitioning plane Π, that

$$\epsilon_1^{\theta,\Pi} = \epsilon_2^{\theta,\Pi} <$$

$$< \max\left\{\epsilon_1^{\theta_i,\Pi}, \epsilon_2^{\theta_i,\Pi}\right\}, \max\left\{\epsilon_1^{\theta_{i+1},\Pi}, \epsilon_2^{\theta_{i+1},\Pi}\right\}.$$

That is, the objective function improves inside the interval $[\theta_i, \theta_{i+1})$ for the angle θ. In fact, this can be happen for a linear number of angles. For example, suppose that p^{θ}_{left} and p^{θ}_{right} are sufficiently far from the Z-axis, and the rest of the elements of L_θ are suffiently close to the Z-axis. Further suppose that the function $d(p^{\theta}_{left}, Z)$ is increasing, and function $d(p^{\theta}_{right}, Z)$ is decreasing in (θ_i, θ_{i+1}), and that they coincide for some some $\theta \in (\theta_i, \theta_{i+1})$. For any bipartition plane Π, we will have that the tolerance functions $\epsilon_1^{\theta,\Pi} \approx d(p^{\theta}_{left}, Z)$ and $\epsilon_2^{\theta,\Pi} \approx d(p^{\theta}_{right}, Z)$ are increasing and decreasing, respectively, and they will also coincide for some angle $\theta \in (\theta_i, \theta_{i+1})$.

Considering all the facts above, we next describe an approximation algorithm running in subquadratic time for solving the 2-fitting problem in 3D, in the case that the orientation of the splitting-plane is fixed. The approximation consists in computing the best bipartition plane for a discrete set of critical angles. That is, we find such a plane for the $O(n)$ angles of the sequence Θ. Our algorithm leaves apart the fact number 4 above, which would imply to consider a quadratic number of critical angles.

2-fitting algorithm in 3D. Fixed orientation of the splitting-plane

1. By Theorems 1 and 2, and Lemma 4, we compute in $O(n \log^2 n)$ time and $O(n \log n)$ space, for all points $p \in P$ the angular intervals $I(p)$ in which p is $\theta_{\{s,t\}}$-active for some $\{s, t\} \in \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}$. We have $O(1)$ intervals for each p, each one associated with the corresponding $\{s, t\}$. For each p, we intersect pairwise the intervals of $I(p)$ to find the set $I'(p)$ of $O(1)$ intervals such that for each interval we have: p is only $\theta_{\{1,2\}}$-active; p is both $\theta_{\{1,2\}}$-active and $\theta_{\{7,8\}}$-active (i.e., p is p^{θ}_{right}); p is only $\theta_{\{5,6\}}$-active; or p is both $\theta_{\{3,4\}}$-active and $\theta_{\{5,6\}}$-active (i.e., p is p^{θ}_{left}).

2. We sort in $O(n \log n)$ time the endpoints of $I'(p)$ for all $p \in P$ to obtain the sequence Θ of the $O(n)$ angles $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_N = \pi$, such that the list L_θ do not change for all $\theta \in [\theta_i, \theta_{i+1})$, $i = 0, 1, \ldots, N - 1$. Thinking on sweeping the sequence Θ with the angle θ from left to right, we associate with each θ_i the point p_i of P and the interval of $I'(p_i)$ with endpoint θ_i. Then, for each θ_i we know which point of P changes some $\theta_{\{s,t\}}$-active condition, and the precise conditions it changes.

3. We sweep Θ from left to right: As a initial step, for $\theta = 0$, we compute the projection of $RCH_0(P)$ on the plane ZY_0, the points p^0_{left} and p^0_{right} in the projection, and build the list L_0 (as a tree) with the $\theta_{\{1,2\}}$-active and $\theta_{\{5,6\}}$-active points in $O(n \log n)$ time.

   In the next steps, for $i = 1, 2, \ldots, N$, we have $\theta = \theta_i$ and we update p^{θ}_{left} and p^{θ}_{right} in constant time from $p^{\theta_{i-1}}_{left}, p^{\theta_{i-1}}_{right}$, and the point p_i associated with θ_i, and update L_θ by inserting/deleting p_i in $O(\log n)$ time. The color of p_i (red or blue) is known according to the $\theta_{\{s,t\}}$-active condition that p_i changes.

   In each step, the initial one and the subsequent ones, we perform the binary search in L_θ in $O(\log^2 n)$ time to find the bipartition plane Π that minimizes $\epsilon_\theta = \max\{\epsilon_1^{\theta,\Pi}, \epsilon_2^{\theta,\Pi}\}$. At the end, we return the angle θ of Θ (joint with its corresponding optimal plane Π) such that ϵ_θ is the smallest over all angles of Θ.

It is clear that the running time of the above algorithm is $O(n \log^2 n)$. We note that the quality of the solution can be improved in terms of ε-approximations. Indeed, for $\varepsilon > 0$, if we split the interval $[0, \pi]$ into sub-intervals of length $\delta = \varepsilon/D$, where D is an upper bound of the absolute value of the first derivative of the functions $\epsilon_1^{\theta,\Pi}$ and $\epsilon_2^{\theta,\Pi}$ for all θ, and apply the binary search also for θ being the endpoints of these sub-intervals, then the solution $APROX$ given by the algorithm is such that $APROX - OPT \leq \delta D$, where OPT denotes the optimal solution. This implies that $OPT \leq APROX \leq OPT + \varepsilon$. The running time will be $O(n \log^2 n + (\pi/\delta) \log^2 n) = O(n \log^2 n + (D\pi/\varepsilon) \log^2 n)$. A value for D can be twice the maximum distance of a point of P to the Z-axis, and can be considered a constant by scaling the point set P. Hence, the final running time is $O(n \log^2 n + \varepsilon^{-1} \log^2 n)$.

Therefore, we arrive to the following theorem:

Theorem 5 *For any $\varepsilon > 0$, an upper bound of the optimal solution of the oriented 2-fitting problem in 3D, with absolute error at most ε, can be obtained in $O(n \log^2 n + \varepsilon^{-1} \log^2 n)$ time and $O(n \log n)$ space if the orientation of the splitting plane is fixed.*

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