Center of maximum-sum matchings of bichromatic points

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Abstract

Let R and B be two disjoint point sets in the plane with $|R| = |B| = n$. Let $\mathcal{M} = \{(r_i, b_i), i = 1, 2, \dots, n\}$ be a perfect matching that matches points of R with points of B and maximizes $\sum_{i=1}^{n} \|r_i - b_i\|$, the total Euclidean distance of the matched pairs. In this paper, we prove that there exists a point o of the plane (the center of $\mathcal{M}$) such that $\|r_i - o\| + \|b_i - o\| \le \sqrt{2}\,\|r_i - b_i\|$ for all $i \in \{1, 2, \dots, n\}$.

1 Introduction

Let R and B be two disjoint point sets in the plane with $|R| = |B| = n$, $n \ge 1$. The points in R are *red*, and those in B are *blue*. A *matching* of $R \cup B$ is a partition of $R \cup B$ into n pairs such that each pair consists of a red and a blue point. A point $p \in R$ and a point $q \in B$ are *matched* if and only if the (unordered) pair (p, q) is in the matching. For every $p, q \in \mathbb{R}^2$, we use pq to denote the segment connecting p and q, and $\|p - q\|$ to denote its length, which is the Euclidean norm of the vector $p - q$. Let $\mathcal{B}(pq)$ denote the disk with diameter equal to $\|p - q\|$, that is centered at the midpoint $\frac{p+q}{2}$ of the segment pq. For any matching $\mathcal{M}$, we use $\mathcal{B}_{\mathcal{M}}$ to denote the set of the disks associated with the matching, that is, $\mathcal{B}_{\mathcal{M}} = \{\mathcal{B}(pq) : (p, q) \in \mathcal{M}\}$.

In this note, we consider the *max-sum* matching $\mathcal{M}$, as the matching that maximizes the total Euclidean distance of the matched points. As our main result, we prove the following theorem:

Theorem 1 *There exists a point o of the plane such that for all $i \in \{1, 2, \dots, n\}$ we have:*

$$\|r_i - o\| + \|b_i - o\| \le \sqrt{2}\,\|r_i - b_i\|.$$

Fingerhut (see Eppstein [3]), motivated by a problem in designing communication networks (see Fingerhut et al. [4]), conjectured that given a set P of $2n$ uncolored points in the plane and a max-sum matching $\{(a_i, b_i), i = 1, \dots, n\}$ of P, there exists a point o of the plane, not necessarily a point of P, such that

$$\|a_i - o\| + \|b_i - o\| \le \frac{2}{\sqrt{3}}\,\|a_i - b_i\| \quad \text{for all } i \in \{1, \dots, n\}, \tag{1}$$

where $2/\sqrt{3} \approx 1.1547$.

Bereg et al. [2] obtained an approximation to this conjecture. They proved that for any point set P of $2n$ uncolored points in the plane and a max-sum matching $\mathcal{M} = \{(a_i, b_i), i = 1, \dots, n\}$ of P, all disks in $\mathcal{B}_{\mathcal{M}}$ have a common intersection, implying that any point o in the common intersection satisfies

$$\|a_i - o\| + \|b_i - o\| \le \sqrt{2}\,\|a_i - b_i\|,$$

where $\sqrt{2} \approx 1.4142$.

Recently, Barabanshchikova and Polyanskii [1] confirmed the conjecture of Fingerhut.

The statement of Equation (1) is equivalent to stating that the intersection $\mathcal{E}(a_1b_1) \cap \mathcal{E}(a_2b_2) \cap \dots \cap \mathcal{E}(a_nb_n)$ is not empty, where $\mathcal{E}(pq)$ is the region of the plane bounded by the ellipse with foci p and q, and major axis length $(2/\sqrt{3})\,\|p - q\|$ (see [3]).

In our context of bichromatic point sets, given $p \in R$ and $q \in B$, let $\mathcal{E}(pq)$ denote the region bounded by the ellipse with foci p and q, and major axis length $\sqrt{2}\,\|p - q\|$. That is, $\mathcal{E}(pq) = \{x \in \mathbb{R}^2 : \|p - x\| + \|q - x\| \le \sqrt{2}\,\|p - q\|\}$. Then, the statement of Theorem 1 is equivalent to stating that the intersection $\mathcal{E}(r_1b_1) \cap \mathcal{E}(r_2b_2) \cap \dots \cap \mathcal{E}(r_nb_n)$ is not empty, for any max-sum matching $\{(r_i, b_i), i = 1, 2, \dots, n\}$ of $R \cup B$.

We note that the factor $\sqrt{2}$ is tight. It suffices to consider two red points and two blue points as vertices of a square, so that each diagonal has vertices of the same color. The center of the square is the only point in common of the two ellipses induced by any max-sum matching.

Hence, to prove Theorem 1 it suffices to consider $n \le 3$, by Helly's Theorem. Let $X_1, X_2, \dots, X_n$ be a collection of n convex subsets of $\mathbb{R}^d$, with $n \ge d + 1$. Helly's Theorem [5] asserts that if the intersection of every $d + 1$ of these subsets is nonempty, then the whole collection has a nonempty intersection. That is why we prove our claim only for $n \le 3$, since we are considering n ellipses in $\mathbb{R}^2$. The arguments that we give in this paper are a simplification and adaptation of the arguments of Barabanshchikova and Polyanskii [1].

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Huemer et al. [6] proved that if $\mathcal{M}'$ is any perfect matching of R and B that maximizes the total *squared* Euclidean distance of the matched points, i.e., it maximizes $\sum_{(p,q)\in\mathcal{M}'} \|p-q\|^2$, then all disks of $\mathcal{B}_{\mathcal{M}'}$ have a point in common. With different techniques, the result of Huemer et al. was extended to higher dimensions by Pirahmad et al. [7]. As proved by Bereg et al. [2], the disks of our max-sum matching $\mathcal{M}$ of $R \cup B$ intersect pairwise, a fact that will be used in this paper, but the common intersection is not always possible.

2 Proof of main result

Let R and B be two disjoint point sets defined as above, where $|R| = |B| = n$, $n \leq 3$, and let $\mathcal{M}$ be a max-sum matching of $R \cup B$. Note that for every pair $(p,q) \in \mathcal{M}$ the disk $\mathcal{B}(pq)$ is inscribed in the ellipse $\mathcal{E}(pq)$ (see Figure 1a), which implies $\mathcal{B}(pq) \subset \mathcal{E}(pq)$. Then, for $n = 2$ Theorem 1 is true because the disks of $\mathcal{M}$ intersect pairwise [2, Proposition 2.1]. Trivially, the theorem is also true for $n = 1$. Therefore, we will prove in the rest of the paper that the theorem is also true for $n = 3$, which will require elaborated arguments.

Let $n = 3$, with $R = \{a, b, c\}$ and $B = \{a', b', c'\}$, and let $\mathcal{M} = \{(a,a'), (b,b'), (c,c')\}$ be a max-sum matching of $R \cup B$.

For two points $p, q \in \mathbb{R}^2$, let $r(pq)$ denote the ray with apex p that goes through q, and for a real number $\lambda \geq 1$, let $\mathcal{E}_\lambda(pq)$ be the region bounded by the ellipse with foci p and q and major axis length $\lambda\|p-q\|$. That is, $\mathcal{E}_\lambda(pq) = \{x \in \mathbb{R}^2 : \|p-x\| + \|q-x\| \leq \lambda\|p-q\|\}$. Note that in our context $\mathcal{E}(pq) = \mathcal{E}_{\sqrt{2}}(pq)$, and $\mathcal{E}_\lambda(pq) \subset \mathcal{E}_{\lambda'}(pq)$ for any $\lambda' > \lambda$.

Assume by contradiction that $\mathcal{E}(aa') \cap \mathcal{E}(bb') \cap \mathcal{E}(cc') = \emptyset$. Then, we can "inflate uniformly" $\mathcal{E}(aa')$, $\mathcal{E}(bb')$, and $\mathcal{E}(cc')$ until they have a common intersection. Formally, we can take the minimum $\lambda > \sqrt{2}$ such that $\mathcal{E}_\lambda(aa') \cap \mathcal{E}_\lambda(bb') \cap \mathcal{E}_\lambda(cc')$ is not empty, which means that $\mathcal{E}_\lambda(aa') \cap \mathcal{E}_\lambda(bb') \cap \mathcal{E}_\lambda(cc')$ is singleton. Let o denote the point of $\mathcal{E}_\lambda(aa') \cap \mathcal{E}_\lambda(bb') \cap \mathcal{E}_\lambda(cc')$.

Let $\ell(aa')$ denote the ray with apex o that bisects $r(oa)$ and $r(oa')$. Similarly, we define $\ell(bb')$ and $\ell(cc')$. Let $t(aa')$ denote the line through o tangent to $\mathcal{E}_\lambda(aa')$, oriented so that $\mathcal{E}_\lambda(aa')$ is to its right. Similarly, we define $t(bb')$ and $t(cc')$. It is well known that given an ellipse with foci p and q, and a line tangent at it at some point o, the rays $r(op)$ and $r(oq)$ form equal angles with the tangent line (see Figure 1b). This implies that rays $\ell(aa')$, $\ell(bb')$, and $\ell(cc')$ are perpendicular to the tangent lines $t(aa')$, $t(bb')$, and $t(cc')$, respectively. In other words, they are contained respectively in the normal lines at point o.

Since $\mathcal{E}(aa')$, $\mathcal{E}(bb')$, and $\mathcal{E}(cc')$ intersect pairwise (and also none of them is contained inside other one), we have that o belongs to the boundary of each of $\mathcal{E}_\lambda(aa')$, $\mathcal{E}_\lambda(bb')$, and $\mathcal{E}_\lambda(cc')$. Then, $\mathcal{E}_\lambda(aa')$, $\mathcal{E}_\lambda(bb')$, and $\mathcal{E}_\lambda(cc')$ intersect pairwise, and each pairwise intersection contains interior points. This implies that no two lines of $t(aa')$, $t(bb')$, and $t(cc')$ coincide. Furthermore, the six directions (positive and negative) of $t(aa')$, $t(bb')$, and $t(cc')$ alternate around o, which implies that any two consecutive rays among $\ell(aa')$, $\ell(bb')$, and $\ell(cc')$ counterclockwise around o, have rotation angle strictly less than π (see Figure 1c).

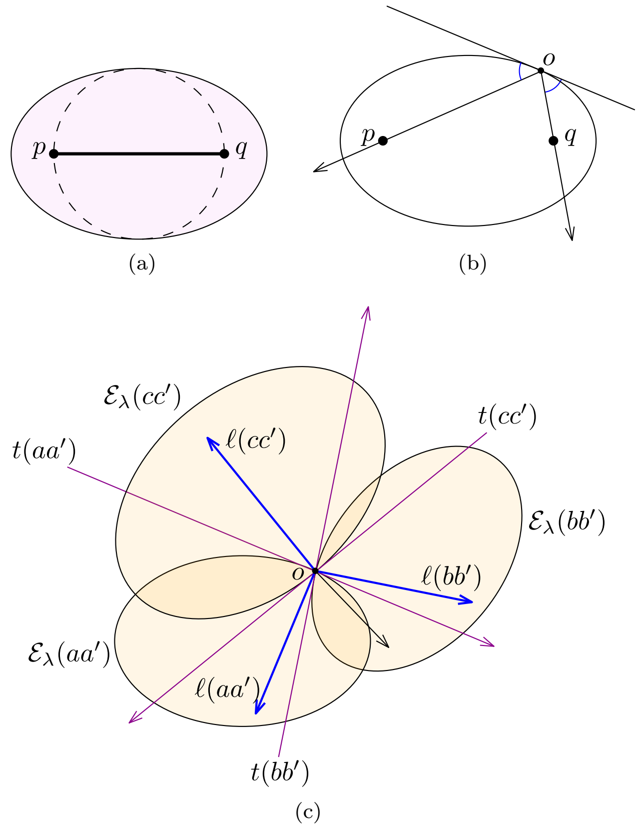

Figure 1: (a) The ellipse $\mathcal{E}(pq)$ and the disk $\mathcal{B}(pq)$. (b) A line tangent to an ellipse forms equal angles with the rays, whose apex is the tangency point, that go through the foci. (c) Point o and the three ellipses.

Let $G = (R \cup B, E)$ be the bipartite graph such that $(p,q) \in E$ if and only if $p \in R$, $q \in B$, and either $(p,q) \in \{(a,a'), (b,b'), (c,c')\}$ or $o \in \mathcal{B}(pq)$. We color the edges into two colors: We say that edge (p,q) is *black* if (p,q) is an edge of the matching, that is, $(p,q) \in \{(a,a'), (b,b'), (c,c')\}$. Otherwise, we say that (p,q) is *white*. Note that this color classification is consistent, since we have that $o \notin \mathcal{B}(pq)$ for all edges $(p,q) \in \{(a,a'), (b,b'), (c,c')\}$ because $\mathcal{B}(pq)$ is contained in the interior of $\mathcal{E}_\lambda(pq)$ and o is in the boundary of $\mathcal{E}_\lambda(pq)$.

The proof of the next lemma is included for completeness.

Lemma 2 ([1]) *If G has a cycle whose edges are color alternating, then $\mathcal{M}$ is not a max-sum matching of*

$R \cup B$.

Proof. For a black edge (p, q) we have that $\|p - o\| + \|q - o\| = \lambda \|p - q\|$. For a white edge (p, q) we have that $\|p - o\| + \|q - o\| < \lambda \|p - q\|$, since $o \in \mathcal{B}(pq)$ and $\mathcal{B}(pq)$ is contained in the interior of $\mathcal{E}_\lambda(pq)$. Let $(r_1, b_1, r_2, b_2, \ldots, r_m, b_m, r_{m+1} = r_1)$ be a cycle of length m, where $r_1, \ldots, r_m \in R$ and $b_1, \ldots, b_m \in B$, and its edges are color alternating. Suppose w.l.o.g. that the edge (r_1, b_1) is black, which means that the edges $(r_1, b_1), \ldots, (r_m, b_m) \in \mathcal{M}$ are all black, and the edges $(b_1, r_2), \ldots, (b_m, r_{m+1}) \in \mathcal{M}$ are all white. Then, we have that:

$$\begin{aligned}\sum_{i=1}^{m} \|r_i - b_i\| &= \frac{1}{\lambda} \sum_{i=1}^{m} (\|r_i - o\| + \|b_i - o\|) \\ &= \frac{1}{\lambda} \sum_{i=1}^{m} (\|b_i - o\| + \|r_{i+1} - o\|) \\ &< \sum_{i=1}^{m} \|b_i - r_{i+1}\|.\end{aligned}$$

Hence, by replacing in $\mathcal{M}$ the black edges of the cycle by the white edges, we will obtain a matching of larger total sum. □

The above alternating cycle idea in the problems about intersections of geometric objects induced by matchings appeared in the proof of Theorem 3 in the paper of Pirahmad et al. [7].

Lemma 3 *Each vertex of G has at least one white edge incident to it.*

Proof. Consider the blue vertex a'. Assume w.l.o.g. that o is the origin of coordinates, and a' is in the positive direction of the y-axis. We have that $\angle aoa' < \pi/2$ because $o \notin \mathcal{B}(aa')$, then assume w.l.o.g. that a is in the interior of the first quadrant Q_1. Let Q_2, Q_3, and Q_4 be the second, third, and fourth quadrants, respectively. Further assume w.l.o.g. that rays $\ell(aa')$, $\ell(bb')$, and $\ell(cc')$ appear in this order counterclockwise.

Assume by contradiction that there is no white edge incident to a'. This implies that b, c belong to the interior of $Q_1 \cup Q_2$. If $c \in Q_2$, then the counterclockwise rotation angle from $\ell(cc')$ to $\ell(aa')$ is larger than π. Hence, $c \in Q_1$. If $b \in Q_1$, then the counterclockwise rotation angle from $\ell(aa')$ to $\ell(bb')$, or that from $\ell(bb')$ to $\ell(cc')$, is larger than π. Hence $b \in Q_2$. Furthermore, if both b' and c' belong to $Q_1 \cup Q_2$, then the counterclockwise rotation angle from $\ell(bb')$ to $\ell(cc')$ is larger than π. Hence, at least one of b', c' belong to the interior of $Q_3 \cup Q_4$. That is, $b' \in Q_3$ and/or $c' \in Q_4$. The proof is divided now into three cases:

Case 1: $b' \in Q_3$ and $c' \in Q_4$. Since $b \in Q_2$ and $c' \in Q_4$, the angle $\angle boc' \geq \pi/2$, which implies that $o \in \mathcal{B}(bc')$ (see Figure 2a). That is, edge (b, c') is

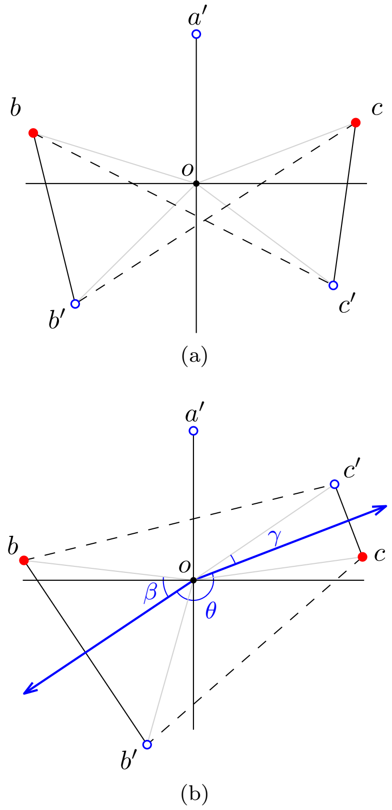

Figure 2: Proof of Lemma 3. Black edges are in normal line style, and white edges in dashed style.

white. Similarly, edge (b', c) is also white. The colors of the edges of the cycle (b, c', c, b', b) alternate, then Lemma 2 implies a contradiction.

Case 2: $b' \in Q_3$ and $c' \notin Q_4$. Since the counterclockwise rotation angle θ from $\ell(bb')$ to $\ell(cc')$ is smaller than π, we must have that $c' \in Q_1$. As in Case 1, we have that edge (b', c) is white, given that $b' \in Q_3$ and $c \in Q_1$. Let β be the half of the angle $\angle bob'$, and γ be the half of the angle $\angle coc'$ (see Figure 2b). Note that $\angle bob' < \pi/2$ and $\angle coc' < \pi/2$ because $o \notin \mathcal{B}(bb')$ and $o \notin \mathcal{B}(cc')$. We have that $\beta, \gamma < \pi/4$, which implies that $\angle boc' \geq 2\pi - \beta - \gamma - \theta \geq \pi/2$. Hence, edge (b, c') is also white. Again, the colors of the edges of the cycle (b, c', c, b', b) alternate, and Lemma 2 implies a contradiction.

Case 3: $b' \notin Q_3$ and $c' \in Q_4$. The proof of this case is analogous to that of Case 2.

The lemma thus follows. □

Lemma 3 implies that the graph G has always a cycle (of length four or six) whose edges are color alternating. Hence, Lemma 2 implies a contradiction, and we obtain that the max-sum matching $\mathcal{M}$ ensures that $\mathcal{E}(aa') \cap \mathcal{E}(bb') \cap \mathcal{E}(cc') \neq \emptyset$. Therefore, Theorem 1 holds.

Acknowledgements: We thank Alexander Polyanskii and Polina Barabanshchikova for sharing and discuss with us their manuscript. P. P-L. was partially supported by project DICYT 041933PL Vicerrectoría de Investigación, Desarrollo e Innovación USACH (Chile). C. S. is supported by project PID2019-104129GB-I00/ MCIN/ AEI/ 10.13039/501100011033.

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