Isomorphisms of simple drawings of complete multipartite graphs

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Simple drawings are drawings of graphs on the sphere in which any two edges intersect at most once (either at a common endpoint or a proper crossing), and no edge intersects itself. When investigating simple drawings, it usually is sufficient to study one representative of each isomorphism class (for different types of isomorphism). One essential element when studying simple drawings are the rotations of crossings or vertices, that is, the cyclic order in which edges emanate from the vertex or crossing. Via rotations and crossings, different types of isomorphisms are defined. Two labeled simple drawings are RS-isomorphic if either all vertices have the same rotations or all have the inverse rotations, CE-isomorphic (also known as weakly isomorphic) if the same pairs of edges cross, CR-isomorphic if either all crossings have the same rotations or all have the inverse rotations, ERS-isomorphic if either all crossings and vertices have the same rotations or all have the inverse rotations, and CO-isomorphic if for each edge the crossings along the edges are in the same order. Finally, two labeled simple drawings are strongly isomorphic if there is a homeomorphism of the sphere that transforms one drawing into the other. Unlabeled simple drawings are isomorphic w.r.t. some type of isomorphism if there exists a labeling such that the labeled drawings are isomorphic w.r.t. that type.

All listed isomorphisms and some combinations of those isomorphisms can be relevant for general graphs. However, there are some isomorphisms implying each other by definition. CR-isomorphism and CO-isomorphism each imply CE-isomorphism, and ERS-isomorphism implies RS-isomorphism and CR-isomorphism (and thus CE-isomorphism). Moreover, for connected graphs, any two simple drawings of the same connected graph are strongly isomorphic if and only if they are ERS-isomorphic and CO-isomorphic (with the same labeling) [1]. For complete graphs, CE-isomorphism, CR-isomorphism, RS-isomorphism, and ERS-isomorphism are all equivalent [2, 3]. Thus, the only relevant types of isomorphism for simple drawings of complete graphs are CE-isomorphism (weak isomorphism) and strong isomorphism.

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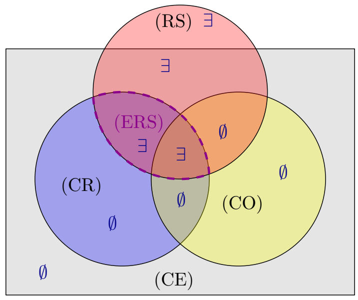

Figure 1: Implications between different isomorphisms for graphs G as defined in Theorem 1. An area is marked with ∃ if there exist drawings of G that are isomorphic w.r.t. exactly the overlapping types of isomorphisms (and no others), and ∅ if there aren't.

As opposed to complete graphs, there are simple drawings of (general) complete multipartite graphs that are RS-isomorphic but not CE-isomorphic. We give a complete characterization which implications do or do not always hold for drawings of complete multipartite graphs, depending on the cardinalities of the partition classes; see Figure 1 for some of our findings. As a main result, we prove the following.

Theorem 1 *Let G be a complete multipartite graph in which each partition class has at least three vertices. Then for any two simple drawings of G it holds that*
(1) CE-isomorphism implies RS-isomorphism, and
(2) CO-isomorphism implies strong isomorphism.

For simple drawings of $K_{2,n}$, we show that RS-isomorphism and CO-isomorphism together imply strong isomorphism, while CE-isomorphism does not imply RS-isomorphism for $n \geq 4$.

References

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