Computing k-Crossing Visibility through k-levels

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1 Introduction

The notion of visibility has been used extensively in Computational Geometry, in the context of the art gallery problem [13, 11]. The development of wireless network connections has motivated the study of a new kind of visibility, where the line of visibility can cross k obstacles [1].

Let $\mathcal{A}$ be an arrangement of straight lines and segments in $\mathbb{R}^2$ (or planes in $\mathbb{R}^3$). The k-crossing visibility on $\mathcal{A}$ of a point p, denoted by $\mathcal{V}_k(p, \mathcal{A})$, is the set of points q on elements of $\mathcal{A}$ such that the segment pq intersects at most k elements of $\mathcal{A}$. See Figure 1.

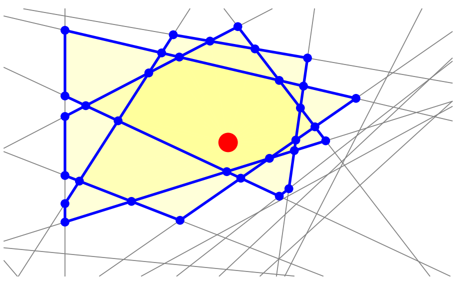

Figure 1: The blue points and segments illustrate the 2-crossing visibility of the red point on an arrangement of lines.

Some early works on k-crossing visibility are [12, 9, 5]. In [4] recently Bahoo et al. introduced an algorithm that computes $\mathcal{V}_k(p, \mathcal{A})$ in $O(kn)$-time, where $\mathcal{A}$ consists of the edges of a polygon.

Theorem 1 (Bahoo et al. [4]) *Given a simple polygon P with n vertices and a query point p in P, the region of P that is k-crossing visible from p, can be computed in $O(kn)$ time.*

In this work we obtain another proof of Theorem 1 and we prove Theorem 2, Theorem 4, Proposition 3 and Proposition 5.

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Theorem 2 *Let $\mathcal{A}$ be an arrangement of n lines in the plane, and let p be a query point. Then $\mathcal{V}_k(p, \mathcal{A})$ can be computed in $O(n \log n + kn)$ time.*

Given an arrangement $\mathcal{A}$ of straight lines, rays and segments in the plane (or planes in $\mathbb{R}^3$), the combinatorial complexity of $\mathcal{A}$, is the total number of vertices and edges (and faces) defined by $\mathcal{A}$.

Proposition 3 *The maximum combinatorial complexity of the k-crossing visibility on arrangements of n straight lines in the plane is $\theta(kn)$.*

Theorem 4 *Let $\mathcal{A}$ be an arrangement of n planes in $\mathbb{R}^3$, and let p be a query point. Then $\mathcal{V}_k(p, \mathcal{A})$ can be computed in $O(n \log n + k^2 n)$ expected time.*

Proposition 5 *The maximum combinatorial complexity of the k-crossing visibility on arrangements of n planes in $\mathbb{R}^3$ is $\theta(k^2 n)$.*

Note that, by Proposition 3 and Proposition 5, Theorem 2 and Theorem 4 are optimal for $k = \Omega(\log n)$ and $k = \Omega(\sqrt{\log n})$, respectively.

Given an arrangement $\mathcal{A}$ of objects in $\mathbb{R}^2$ (or $\mathbb{R}^3$), the $(\leq k)$-level-region of $\mathcal{A}$ is the set of points in $\mathbb{R}^2$ (or $\mathbb{R}^3$) with at most k elements of $\mathcal{A}$ lying above it. In the following we denote by $(\leq k)level(\mathcal{A})$, to the portion of the elements of $\mathcal{A}$ that are in the $(\leq k)$-level-region of $\mathcal{A}$.

Let $\mathcal{T}$ be the transformation

$$\mathcal{T}((x,y)) = (x/y, 1/y) \qquad \text{in the } \mathbb{R}^2 \text{ case, or}$$
$$\mathcal{T}((x,y,z)) = (x/z, y/z, 1/z) \qquad \text{in the } \mathbb{R}^3 \text{ case.}$$

In this paper we obtain a linear time reduction, of the problem of obtaining $\mathcal{V}_k(p, \mathcal{A})$ to the problem of obtaining $(\leq k)level(\mathcal{A})$, by applying $\mathcal{T}$. This reduction can be easily adapted for obtaining k-crossing visibilities on another arrangements whose $(\leq k)$-level is known.

2 Results in $\mathbb{R}^2$

Let $\mathcal{D}$ be the set of points $(x, y) \in \mathbb{R}^2$ such that $y \neq 0$. Throughout this section, O denotes the point $(0, 0) \in \mathbb{R}^2$ and $\mathcal{T}$ denotes the transformation $\mathcal{T} : \mathcal{D} \to \mathcal{D}$ such that $\mathcal{T}((x, y)) = (x/y, 1/y)$.

In this section, we prove that $\mathcal{T}$ determines a bridge between k-crossing visibility and $(\leq k)$-levels in $\mathbb{R}^2$. Then, we use this result to prove Theorem 1, Theorem 2 and Proposition 3.

2.1 Properties of $\mathcal{T}$

Given $D \subset \mathcal{D}$ we denote by $\mathcal{T}[D]$ the image of D under $\mathcal{T}$. We also denote by $\mathcal{T}[\mathcal{A}]$ the set images of the elements of $\mathcal{A}$ under $\mathcal{T}$. In this section, we first prove several properties of $\mathcal{T}$. Then, we determine $\mathcal{T}[D]$ for different instances of D. Finally, we prove that $\mathcal{V}_k(O, \mathcal{A})$ can be obtained from $(\leq k)level(\mathcal{T}[\mathcal{A}])$.

Proposition 6 *$\mathcal{T}$ is self-inverse.*

Proof. $\mathcal{T} o \mathcal{T}((x, y)) = \mathcal{T}((x/y, 1/y)) = (x, y)$. ☐

Proposition 7 *$\mathcal{T}$ sends straight lines to straight lines. More precisely, if L is the straight line in $\mathcal{D}$ with equation $ax + by + c = 0$ then $\mathcal{T}[L]$ is the straight line in $\mathcal{D}$ with equation $ax + cy + b = 0$.*

Proof. Let L' be the straight line with equation $ax + cy + b = 0$. If (x_0, y_0) is in L then $ax_0 + by_0 + c = 0$; thus, as $a\frac{x_0}{y_0} + c\frac{1}{y_0} + b = 0$, then $\mathcal{T}(x_0, y_0)$ is in L'. Similarly, if (x_0, y_0) is in L' then $\mathcal{T}^{-1}(x_0, y_0) = \mathcal{T}(x_0, y_0)$ is in L. ☐

Proposition 8 *$\mathcal{T}$ preserve incidences between points and lines. More precisely the point p is in the straight line L if and only if $\mathcal{T}(p)$ is in the straight line $\mathcal{T}[L]$.*

Proof. Let $p = (x_0, y_0)$ be a point in $\mathcal{D}$ and let $L : ax + by + c = 0$ be a straight line in $\mathcal{D}$. This proof follows from the fact that (x_0, y_0) satisfies $ax + by + c = 0$ if and only if $\left(\frac{x_0}{y_0}, \frac{1}{y_0}\right)$ satisfies $\mathcal{T}[L] : ax + cy + b = 0$. ☐

Given a line L in $\mathcal{D}$, we denote by L^+ the set of points in L whose second coordinate is greater than zero, and we denote by L^- the set of points in L whose second coordinate is less than zero.

Proposition 9 *Let L be a straight line in $\mathcal{D}$. Then $\mathcal{T}[L^+] = \mathcal{T}[L]^+$ and $\mathcal{T}[L^-] = \mathcal{T}[L]^-$. Moreover, If $p_1, p_2, \ldots, p_k$ are in L^+ (or they are in L^-) ordered by their distance to the x-axis from the closest to the furthest, then $\mathcal{T}(p_1), \mathcal{T}(p_2), \ldots, \mathcal{T}(p_k)$ are in $\mathcal{T}[L^+]$ (or they are in $\mathcal{T}[L^-]$, respectively), ordered by their distance to the x-axis from the furthest to the closest.*

Proof. As $\mathcal{T}$ sends straight lines to straight lines and it does not change the sign of the second coordinate, then $\mathcal{T}[L^+] = \mathcal{T}[L]^+$ and $\mathcal{T}[L^-] = \mathcal{T}[L]^-$. If the second coordinates of p_i and p_j are y_i and y_j, respectively, then the second coordinates of $\mathcal{T}(p_i)$ and $\mathcal{T}(p_j)$ are $1/y_i$ and $1/y_j$, respectively. This proof follows from the fact that $|y_i| < |y_j|$ if and only if $|1/y_i| > |1/y_j|$. ☐

Let $\mathcal{D}^+$ denote the set of points in $\mathcal{D}$ whose second coordinate is greater than zero, and let $\mathcal{D}^-$ denote the set of points in $\mathcal{D}$ whose second coordinate is less than zero. The proofs of Proposition 10 and Proposition 11 follows from Proposition 9.

Proposition 10 *Let D be a line segment contained in a straight line L, whose endpoints are p and q.*

- *If both p and q are in $\mathcal{D}^+$ ($\mathcal{D}^-$), then $\mathcal{T}[D]$ is the line segment contained in $\mathcal{D}^+$ ($\mathcal{D}^-$) whose endpoints are $\mathcal{T}(p)$ and $\mathcal{T}(q)$.*
- *If p is in the x-axis and q is in $\mathcal{D}^+$ ($\mathcal{D}^-$), then $\mathcal{T}[D]$ is the ray contained in $\mathcal{D}^+$ ($\mathcal{D}^-$), defined by the straight line $\mathcal{T}[L]$ and the point $\mathcal{T}(q)$.*

Given a $D \subset \mathcal{D}$ we denote by $\overline{D}$ the closure of D in $\mathbb{R}^2$.

Proposition 11 *Let D be a no horizontal ray contained in a straight line L, whose endpoint is p.*

- *If both p and D are in $\mathcal{D}^+$ ($\mathcal{D}^-$), then $\mathcal{T}[D]$ is the line segment contained in $\mathcal{D}^+$ ($\mathcal{D}^-$), whose endpoints are $\mathcal{T}(p)$ and the intersection of $\overline{\mathcal{T}[L]}$ with the x-axis.*
- *If p is in the x-axis and D is in $\mathcal{D}^+$ ($\mathcal{D}^-$), then $\mathcal{T}[D]$ is the ray defined by the part of the straight line $\mathcal{T}[L]$ in $\mathcal{D}^+$ ($\mathcal{D}^-$).*

Proposition 12 *Let D be a horizontal ray contained in a straight line L whose endpoint is p. If D is contained in $\mathcal{D}^+$ ($\mathcal{D}^-$), then $\mathcal{T}[D]$ is the horizontal ray in $\mathcal{D}^+$ ($\mathcal{D}^-$), defined by the straight line $\mathcal{T}[L]$ and the point $\mathcal{T}(p)$. If D is contained in $\mathcal{D}^+$, then D and $\mathcal{T}[D]$ have the same direction; in the other case, D and $\mathcal{T}[D]$ have opposite direction.*

Proof. If L has equation $by + c = 0$ then $\mathcal{T}[L]$ is the horizontal line with equation $cy + b = 0$. ☐

From Proposition 7, Proposition 10, Proposition 11 and Proposition 12 we conclude that: If D is a straight line, ray or segment contained in $\mathcal{D}^+$ ($\mathcal{D}^-$) then $\mathcal{T}[D]$ is a straight line, ray or segment contained in $\mathcal{D}^+$ ($\mathcal{D}^-$).

Proposition 13 *Let L be a straight line in $\mathcal{D}$. Then $O \in \overline{L}$ if and only if $\mathcal{T}[L]$ is a vertical line.*

Proof. This proof follows from the fact that $\overline{L}$ has equation $ax + by = 0$ if and only if $\mathcal{T}[L]$ has equation $ax + b = 0$ ☐

Let $\mathcal{V}_k^+(O, \mathcal{A})$ denote the portions of the elements of $\mathcal{V}_k(O, \mathcal{A})$ in $\mathcal{D}^+$, *i.e.*

$$\mathcal{V}_k^+(O, \mathcal{A}) = \left\{D \cap \mathcal{D}^+ : D \in \mathcal{V}_k(O, \mathcal{A})\right\}$$

Similarly, let $\mathcal{V}_k^-(O,\mathcal{A})$ denote the portions of $\mathcal{V}_k(O,\mathcal{A})$ in $\mathcal{D}^-$, *i.e.*

$$\mathcal{V}_k^-(O,\mathcal{A}) = \left\{D \cap \mathcal{D}^- : D \in \mathcal{V}_k(O,\mathcal{A})\right\}$$

Let $(\leq k)level^+(\mathcal{A})$ denote the portion of the elements of $(\leq k)level(\mathcal{A})$ in $\mathcal{D}^+$, *i.e.*

$$(\leq k)level^+(\mathcal{A}) = \left\{D \cap \mathcal{D}^+ : D \in (\leq k)level(\mathcal{A})\right\}$$

The $(\leq k)$-lower-level-region of $\mathcal{A}$ is the set of points in $\mathbb{R}^2$ ($\mathbb{R}^3$) with at most k elements of $\mathcal{A}$ lying below it. Let $(\leq k)level^-(\mathcal{A})$ denote the portion of the elements of $\mathcal{A}$ in both $\mathcal{D}^-$ and the $(\leq k)$-lower-level-region of $\mathcal{A}$.

Lemma 14 *Let $\mathcal{A}$ be an arrangement of straight lines, segments or rays. Then:*

1. $\mathcal{V}_k^+(O,\mathcal{A}) = \mathcal{T}\left[(\leq k)level^+(\mathcal{T}[\mathcal{A}])\right]$.
2. $\mathcal{V}_k^-(O,\mathcal{A}) = \mathcal{T}\left[(\leq k)level^-(\mathcal{T}[\mathcal{A}])\right]$.

Proof. We prove 1, the proof of 2 is similar.

Let $p \in \mathcal{D}^+$ be such that $p \in D$ for some $D \in \mathcal{A}$, and let L be the line that contains p and O. Then $p \in L^+$, $\mathcal{T}[D] \in \mathcal{T}[\mathcal{A}]$ and $\mathcal{T}(p) \in \mathcal{T}[D]$. As $\mathcal{T}$ preserves incidences, by Proposition 13 and Proposition 9, the line segment between O and p crosses at most k elements of $\mathcal{A}$, if and only if, there are at most k elements of $\mathcal{T}[\mathcal{A}]$ laying above $\mathcal{T}(p)$. ☐

2.2 Proofs of results in $\mathbb{R}^2$

Theorem 15 (Everett et al. [10]) *Let $\mathcal{A}$ be an arrangement of n lines in the plane. Then $(\leq k)level(\mathcal{A})$ can be computed in $O(n \log n + kn)$ time.*

We use Theorem 15 in order to prove Theorem 2.

Proof. [Proof of Theorem 2] Without loss of generality, we may assume that p is at the origin, otherwise p and the elements of $\mathcal{A}$ can be translated. We also may assume that the x-axis does not contain an element of $\mathcal{A}$ or an intersection between two elements of $\mathcal{A}$, otherwise, the elements of $\mathcal{A}$ can be rotated.

By Proposition 7, $\mathcal{T}[\mathcal{A}]$ is an arrangement of n straight lines. Thus, as the k-crossing visibility of O on $\mathcal{A}$ can be obtained from $\mathcal{V}_k^+(O,\mathcal{A})$ and $\mathcal{V}_k^-(O,\mathcal{A})$, this proof follows from Lemma 14 and Theorem 15. ☐

Let $\mathcal{A}$ be an arrangement of straight lines, rays and segments. The vertical decomposition (also known as trapezoidal decomposition) of $\mathcal{A}$ is obtained by erecting vertical segments upwards and downwards from each vertex in $\mathcal{A}$ and extend them until they meet another line or all the way to infinity.

Lemma 16 *Let $\mathcal{A}$ be an arrangement of n straight lines, rays and segments. Then $(\leq k)level(\mathcal{A})$ can be obtained from a vertical decomposition of $\mathcal{A}$ in $O(kn)$ time.*

Proof. Suppose that the vertical decomposition of A is known. Then for each vertex, extend a vertical segment upwards until it reaches $k+1$ elements of $\mathcal{A}$ or its way to infinity; such vertex is in $(\leq k)level(\mathcal{A})$ if and only if the vertical segment reaches its way to infinity. ☐

Proof. [Another proof of Theorem 1] As in the proof of Theorem 2, we may assume that p is at the origin and the x-axis does not contain edges of P. By Proposition 10, $\mathcal{T}[P]$ is an arrangement of at most $2n$ line segments or rays. Thus, as the k-crossing visibility of O on P can be obtained from $\mathcal{V}_k^+(O,P)$ and $\mathcal{V}_k^-(O,P)$, by Lemma 14 and Lemma 16, it is enough to obtain the vertical decomposition of $\mathcal{T}[P] \cap \mathcal{D}^+$ and $\mathcal{T}[P] \cap \mathcal{D}^-$ in linear time; we do this for $\mathcal{T}[P] \cap \mathcal{D}^+$, the other case is similar.

Let $L : y + c = 0$ be a horizontal line, high enough that all the endpoints of $\mathcal{T}[P] \cap \mathcal{D}^+$ are below L. Let $L' = \mathcal{T}[L]$ and note that L' is a horizontal line with equation $cy + 1 = 0$. Suppose that the points in P above L' are blue and the others are red. Let P' be the polygon in $\mathcal{D}^+$ obtained from P by scaling vertically its red part, keeping the endpoints on L' fixed.

In [7] Chazelle prove that the vertical decomposition of a polygon can be computed in linear time (see also Amato et al. [3]). Thus, as P' is contained in $\mathcal{D}^+$, by Proposition 10 $\mathcal{T}[P']$ is a polygon, and the vertical decomposition of $\mathcal{T}[P']$ can be computed in linear time. Note that a point $\mathcal{T}(q)$ in $\mathcal{T}[P] \cap \mathcal{D}^+$ is below L if and only if q is blue. Thus the vertical decomposition of $\mathcal{T}[P] \cap \mathcal{D}^+$ can be obtained from the vertical decomposition of $\mathcal{T}[P']$ below L. ☐

Proof. [Proof of Proposition 3] In [2] Alon et al. prove that the maximum combinatorial complexity of the $(\leq k)$-level on arrangements of n straight lines in the plane is $\theta(nk)$. Without loss of generality, suppose that the arrangement $\mathcal{A}$ reaches this bound and the $(\leq k)$-level of $\mathcal{A}$ is contained in $\mathcal{D}^+$. Thus, the combinatorial complexity of $(\leq k)level^+(\mathcal{A})$ is $\theta(nk)$ and by Lemma 14 the combinatorial complexity of $\mathcal{V}_k^+(O,\mathcal{T}[\mathcal{A}])$ is also $\theta(nk)$. ☐

3 Results in $\mathbb{R}^3$

Let $\mathcal{D}$ be the set of points $(x,y,z) \in \mathbb{R}^3$ such that $z \neq 0$. Throughout this section, O denotes the point $(0,0,0) \in \mathbb{R}^3$ and $\mathcal{T}$ denotes the transformation $\mathcal{T} : \mathcal{D} \to \mathcal{D}$ be such that

$$\mathcal{T}((x,y,z)) = (x/z, y/z, 1/z).$$

The proofs of Proposition 17, Proposition 18, Proposition 19, Proposition 20, Proposition 21 and Lemma 22, can be obtained as in Section 2.

Proposition 17 *$\mathcal{T}$ is self-inverse.*

Proposition 18 *$\mathcal{T}$ sends planes to planes. More precisely, if π is the plane in $\mathcal{D}$ with equation $ax + by + cz + d = 0$ then $\mathcal{T}[\pi]$ is the plane in $\mathcal{D}$ with equation $ax + by + dz + c = 0$.*

Proposition 19 *$\mathcal{T}$ preserve incidences between points and planes. More precisely the point p is in the plane π if and only if $\mathcal{T}(p)$ is in the plane $\mathcal{T}[\pi]$.*

Given a plane π in $\mathcal{D}$, we denote by π^+ the set of points in π whose third coordinate is greater than zero, and we denote by π^- the set of points in π whose third coordinate is less than zero.

Proposition 20 *Let π be a plane in $\mathcal{D}$. Then $\mathcal{T}[\pi^+] = \mathcal{T}[\pi]^+$ and $\mathcal{T}[\pi^-] = \mathcal{T}[\pi]^-$. Moreover, If $p_1, p_2, \ldots, p_k$ are in π^+ (or they are in π^-) ordered by their distance to the plane $z = 0$ from the closest to the furthest, then $\mathcal{T}(p_1), \mathcal{T}(p_2), \ldots, \mathcal{T}(p_k)$ are in $\mathcal{T}[\pi^+]$ (or they are in $\mathcal{T}[\pi^-]$, respectively), ordered by their distance to the plane $z = 0$ from the furthest to the closest.*

Given a $D \subset \mathcal{D}$ we denote by $\overline{D}$ the closure of D in $\mathbb{R}^3$.

Proposition 21 *Let L be a straight line in $\mathcal{D}$. Then $O \in \overline{L}$ if and only if $\mathcal{T}[L]$ is a vertical line.*

Lemma 22 *Let $\mathcal{A}$ be an arrangement of planes. Then:*

1. $\mathcal{V}_k^+(O, \mathcal{A}) = \mathcal{T}[(\leq k)level^+(\mathcal{T}[\mathcal{A}])]$.
2. $\mathcal{V}_k^-(O, \mathcal{A}) = \mathcal{T}[(\leq k)level^-(\mathcal{T}[\mathcal{A}])]$.

The proofs of Theorem 4 and Proposition 5 follows from Theorem 23 and Theorem 24, in a similar way as in the proof of Theorem 2 and the proof of Proposition 3 in Section 2.

Theorem 23 (Chan [6]) *Let $\mathcal{A}$ be an arrangement of n planes in $\mathbb{R}^3$. Then $(\leq k)level(\mathcal{A})$ can be computed in $O(n \log n + k^2 n)$ expected time.*

Theorem 24 (Clarkson et al. [8]) *Let $k \geq 1$. Then the maximum combinatorial complexity of $(\leq k)$-level on arrangements of n hyperplanes in $\mathbb{R}^d$ is $\theta\left(n^{\lfloor d/2 \rfloor} k^{\lceil d/2 \rceil}\right)$.*

References

[1] Oswin Aichholzer, Ruy Fabila-Monroy, David Flores-Peñaloza, Thomas Hackl, Jorge Urrutia, and Birgit Vogtenhuber. Modem illumination of monotone polygons. *Computational Geometry*, 68:101–118, 2018.

[2] Noga Alon and Ervin Györi. The number of small semispaces of a finite set of points in the plane. *Journal of Combinatorial Theory, Series A*, 41(1):154–157, 1986.

[3] Nancy M Amato, Michael T Goodrich, and Edgar A Ramos. A randomized algorithm for triangulating a simple polygon in linear time. *Discrete & Computational Geometry*, 26(2):245–265, 2001.

[4] Yeganeh Bahoo, Prosenjit Bose, Stephane Durocher, and Thomas Shermer. Computing the k-crossing visibility region of a point in a polygon. In Charles J. Colbourn, Roberto Grossi, and Nadia Pisanti, editors, *Combinatorial Algorithms*, pages 10–21, Cham, 2019. Springer International Publishing.

[5] Antonio Bajuelos, Santiago Canales, Gregorio Hernández, and Mafalda Martins. A hybrid metaheuristic strategy for covering with wireless devices. *J. Univers. Comput. Sci.*, 18(14):1906–1932, 2012.

[6] Timothy Chan. Random sampling, halfspace range reporting, and construction of $(\leq k)$-levels in three dimensions. *SIAM Journal on Computing*, 30(2):561–575, 2000.

[7] Bernard Chazelle. Triangulating a simple polygon in linear time. *Discrete & Computational Geometry*, 6(3):485–524, 1991.

[8] Kenneth Clarkson and Peter Shor. Applications of random sampling in computational geometry, ii. *Discrete & Computational Geometry*, 4(5):387–421, 1989.

[9] James Dean, Andrzej Lingas, and Jörg-Rüdiger Sack. Recognizing polygons, or how to spy. *The Visual Computer*, 3(6):344–355, 1988.

[10] Hazel Everett, Jean-Marc Robert, and Marc van Kreveld. An optimal algorithm for the $(\leq k)$-levels, with applications to separation and transversal problems. *International Journal of Computational Geometry Applications*, 06(03):247–261, 1996.

[11] Subir Kumar Ghosh. *Visibility algorithms in the plane*. Cambridge university press, 2007.

[12] Naji Mouawad and Thomas Shermer. The superman problem. *The Visual Computer*, 10(8):459–473, 1994.

[13] Joseph O'Rourke et al. *Art gallery theorems and algorithms*, volume 57. Oxford University Press Oxford, 1987.