On the Realizability of Planar and Spherical Occlusion Diagrams

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Abstract

Spherical occlusion diagrams (SOD) were introduced as an axiomatic framework to analyze the visibility maps of points in the interior of a nonconvex polyhedron from which no vertex is visible. Planar occlusion diagrams (POD), corresponding to a viewpoint at infinity, can be defined analogously in terms of orthogonal projections of a lower envelope of polyhedra. We have recently constructed PODs and SODs that are not realizable as visibility maps. Here we show that every axis-aligned POD is realizable and follow up with open problems.

Introduction. The classical *art gallery problem* asks for the minimum number of point guards that can jointly see all points in a nonconvex polyhedron $\mathcal{P}$ in Euclidean space, where points s and t *see* each other if the line segment st does not cross any face of $\mathcal{P}$. It is well known that guards stationed at the vertices of $\mathcal{P}$ do not always suffice in $\mathbb{R}^3$, as some points $s \in \mathbb{R}^3$ in the interior of a polyhedron $\mathcal{P}$ may not see any of the vertices [4, Sec. 10.2]. Viglietta [7] recently introduced *spherical occlusion diagrams* (*SOD*, for short) to analyze the visibility map $V_{\mathcal{P}}(s)$ of such a point s with respect to $\mathcal{P}$. An SOD is defined so that it satisfies key properties of visibility maps. In particular, if no vertices of a polyhedron $\mathcal{P}$ are visible from a viewpoint s, then the visibility map $V_{\mathcal{P}}(s)$ is an SOD. Viglietta conjectured that the converse also holds, that is, every SOD is the visibility map $V_{\mathcal{P}}(s)$ for some point s and polyhedron $\mathcal{P}$ in $\mathbb{R}^3$. We have recently disproved this conjecture, by constructing an SOD that is not realizable as a visibility map in $\mathbb{R}^3$ [3].

Related work. Our results show that SODs are not always visibility maps. Nevertheless, SODs have already been used in 3-dimensional visibility problems: Cano et al. [2] proved that every polyhedron $\mathcal{P}$ in $\mathbb{R}^3$ can be guarded by at most $\frac{5}{6}$ of its edges; moreover, when $\mathcal{P}$ is homeomorphic to a ball and all its faces are triangles, it can be guarded by at most $\frac{29}{36}$ of its edges. Tóth et al. [6] proved that every point that does not see any vertex of a polyhedron $\mathcal{P}$ must see at least 8 edges of $\mathcal{P}$, and this bound is tight.

The realizability of visibility maps has been previously studied for lines. A *weaving pattern* is a simple arrangement of n lines in $\mathbb{R}^2$ together with a binary relation between intersecting lines; a weaving pattern is *realizable* if it is the orthogonal projection of an arrangement of disjoint lines in $\mathbb{R}^3$ such that the above-below relation between lines matches the given binary relation between their orthogonal projections. Pach et al. [5] showed that almost all weaving patterns of n lines are nonrealizable for sufficiently large n. Basu et al. [1] generalized the result to arrangements of semi-algebraic curves.

Outlook. We have shown that spherical occlusion diagrams (SODs) are not equivalent to visibility maps in 3-space. Our result raises several open problems: Is there a simple (axiomatic) characterization of visibility maps? Can one decide efficiently whether a given SOD is a visibility map? If so, can one find a realization efficiently? What is the maximum (combinatorial, topological, or bit) complexity of the realization space for an SOD with n arcs for a given positive integer n?

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