# Computing the $(\alpha, k)$-hull for points in convex position

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## Abstract

We present an efficient $O(n \log n)$-time and $O(n)$-space algorithm for computing the $(\alpha, k)$-hull of a set $P$ of $n$ points in convex position in the plane.

## 1 Introduction

Let $P$ be a set of $n \geq 3$ points in convex position in the plane, $\alpha \in (0, \pi]$ an angle, and $k \in \mathbb{N}$, $1 \leq k \leq \lfloor n/2 \rfloor$. The $(\alpha, k)$-hull of the set $P$ is the (possible empty) curvilinear region defined by the intersection of all $\alpha$-halfplanes that contain at least $n - k + 1$ points of $P$, where an $\alpha$-halfplane is the complement of an open wedge of aperture angle $\alpha$. That is, the $(\alpha, k)$-hull of $P$ is the locus of the points $u$ such that any wedge with apex $u$ and aperture angle $\alpha$ contains at least $k$ points of $P$. See Claverol et al. [1], where an $O(n^2 \log n)$-time algorithm was presented for computing this hull.

The $(\alpha, k)$-hull of a point set has tentative real applications. Namely, suppose that $P$ represents a set of key points of a terrain, and we need to install a surveillance camera which has an angle of vision equal to $\alpha$, rotates constantly, and at every moment should watch at least $k$ of the key points. Hence, the $(\alpha, k)$-hull of $P$ is the locus of the possible locations in the terrain to install the rotating camera.

## 2 The algorithm

Each $k+1$ consecutive vertices of $conv(P)$ define an open disk $D$ such that all points of $conv(P) \setminus D$ see these vertices with angle at least $\alpha$. We have $n$ disks in total, and it can be proved that $conv(P) \setminus \mathcal{D}$ is precisely the $(\alpha, k)$-hull of $P$, where $\mathcal{D}$ is the union of all disks $D$.

The contour of $\mathcal{D}$ consists of $O(n)$ circular arcs [2], and by using the algorithm of Imai et al. [2] based on a Voronoi diagram for disks, they can be computed in $O(n \log n)$ time. The interior contour of $\mathcal{D}$, bounding the holes of $\mathcal{D}$, defines the (connected components of) the $(\alpha, k)$-hull of $P$. See Figure 1 for a couple of examples of the $(\alpha, k)$-hull of $P$. Thus, we have the following result.

**Theorem 1** *Given $\alpha \in (0, \pi]$ and $1 \leq k \leq \lfloor n/2 \rfloor$, the $(\alpha, k)$-hull of a set $P$ of $n$ points in convex position has $O(n)$ complexity, and can be computed in efficient $O(n \log n)$ time and $O(n)$ space.*

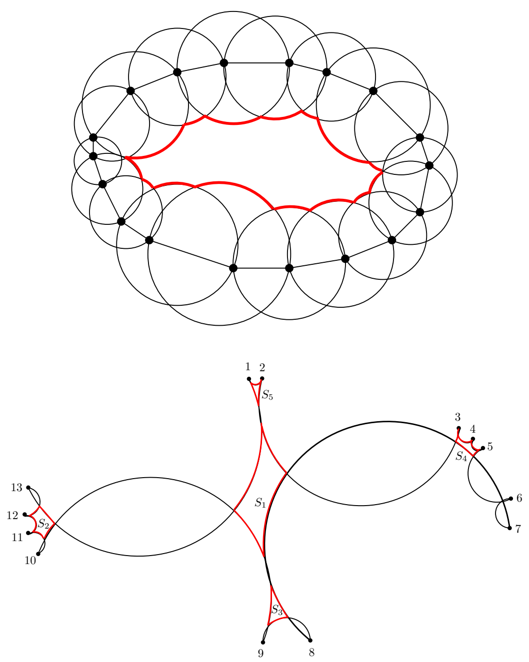


Figure 1: (Top) The $(\pi/2, 2)$-hull formed by a connected region. (Bottom) The $(\pi/2, 1)$-hull formed by 5 connected regions.

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## References

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[2] H. Imai, M. Iri, and K. Murota. Voronoi diagram in the Laguerre geometry and its applications. *SIAM J. Comput.*, **14**, (1985), pp. 93–105.