Fault-tolerant resolvability in maximal outerplanar graphs

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Resolving sets are useful to distinguish the vertices of a graph. A set S of vertices of a graph G is a *resolving set* if for every pair of vertices u and v of G there is at least one vertex w in S such that the distances from w to u and from w to v are distinct. The *metric dimension* of G, denoted by $\beta(G)$, is the minimum cardinality of a resolving set. These concepts were introduced for general graphs independently by Slater [4] and by Harary and Melter [2], and have since been widely investigated.

Fault-tolerant resolving sets were introduced to distinguish the vertices of a graph even if a vertex fails [3]. A resolving set S of a non-trivial connected graph G is *fault-tolerant* if $S \setminus \{v\}$ is also a resolving set for each $v \in S$. The *fault-tolerant metric dimension* of G, denoted by $\beta'(G)$, is the minimum cardinality of a fault-tolerant resolving set of G. Since $V(G)$ and $V(G) \setminus \{v\}$ are both resolving sets for every vertex v of a graph G of order at least 2, this parameter is well-defined whenever G is a non-trivial graph. Moreover, $\beta'(G) \leq n$ and, obviously, $\beta'(G) \geq \beta(G) + 1$.

A planar graph is *outerplanar* if it admits a plane embedding such that all the vertices belong to the unbounded face. A *maximal outerplanar* graph is an outerplanar graph such that the addition of an edge results in a non-outerplanar graph. Maximal outerplanar graphs can be viewed as triangulated polygons.

This ongoing work is devoted to studying the fault-tolerant resolvability for maximal outerplanar graphs. Our first goal is to prove lower and upper bounds on the fault-tolerant metric dimension. Hence, the study of resolving sets and the metric dimension of maximal outerplanar graphs will be useful for our purpose. In [1], the authors prove that the metric dimension of a maximal outerplanar graph of order n, $n \geq 5$, is at most $\lceil \frac{2n}{5} \rceil$. Moreover, this bound is tight and is attained for some fan graphs, a special family of maximal outerplanar graphs.

It is easy to see that $\beta'(G) \geq 3$, if G is a maximal outerplanar graph. Furthermore, we show that only two maximal outerplanar graphs, having orders 3 and 6, attain this lower bound. For maximal outerplanar graphs of order at least 7, the lower bound for $\beta'(G)$ is 4 and there is an infinite family of maximal outerplanar graphs such that $\beta(G) = 2$ attaining this bound.

Regarding to the upper bound, we conjecture that $\beta'(G) \leq \lceil \frac{n}{2} \rceil$ for a maximal outerplanar graph G of order n, $n \geq 7$. At the moment, we have proved that fan graphs attain this upper bound. Moreover, fan graphs of even order have only one fault-tolerant resolving set of minimum size, concretely, the set formed by alternating vertices of the unbounded face and not containing the vertex of degree $n-1$ (see Figure 1).

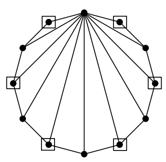

Figure 1: The only fault-tolerant resolving set of size 6 of the fan of order 12 is formed by the squared vertices.

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