Special constructions to understand the structure of higher order Voronoi diagrams

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Abstract

We study the structure of faces in Voronoi diagrams of order k, $V_k(S)$, of sets S of n points in the plane. While the number of faces of $V_k(S)$ is well known, not so much is known about the numbers of quadrilaterals, of pentagons, of hexagons in $V_k(S)$. We present two extremal point sets and calculate the number of faces of each type in $V_k(S)$. Among the obtained results, we show that there exists a set S of n points, where all bounded faces of $V_k(S)$ are hexagons, for $k \geq (n+3)/2$, and where $V_k(S)$ contains no quadrilateral for $3 \leq k \leq (n+1)/4$. Finally, we prove that for no point set S, $V_k(S)$ can have two adjacent quadrilaterals, for $k \geq 2$, and we present some experimental result.

1 Introduction

We present a study on higher order Voronoi diagrams, that continues our previous work [3]. Voronoi diagrams are a very useful tool in diverse disciplines, see e.g. [1, 11]. Many of their properties were already obtained by Lee [7]. For a given set S of n points in general position in the plane, meaning that no three points of S are collinear and no four points of S are cocircular, the Voronoi diagram of order k of S, $V_k(S)$, is a subdivision of the plane into faces such that points in the same face have the same k nearest points of S. A face of $V_k(S)$ is denoted by $f(P_k)$ where P_k is the subset of k points of S that is closest to every point of this face. It is well known that $V_k(S)$ has $(2k-1)n-(k^2-1)-\sum_{j=0}^{k-2} e_j$ many faces, see e.g. [7, 3]. Here, e_j denotes the number of j-edges of S. A j-edge is a half-plane defined by the oriented line through a pair of points of S that contains j points of S in its interior. The set P_k associated to an unbounded face can be separated from $S \setminus P_k$ by a straight line, and the number of unbounded faces of $V_{k+1}(S)$ is e_k.

Miles and Maillardet [9] proved that $V_k(S)$ never contains a triangle for $k \geq 2$, also see [3, 8]. We are interested in the number of quadrilateral faces, of pentagonal faces, etc., of $V_k(S)$. This question has been studied extensively for $k = 1$ and for random point sets, especially with respect to a homogeneous Poisson point process, see e.g. [2, 4, 10, 6]. Several of these results are experimental and are summarized in [11]. In this setting, the expected number of sides of a face of $V_k(S)$ is 6 for every $1 \leq k \leq n-2$ [10]. In order to study how many faces with i sides, for $i = 4, 5, \ldots$, are there at least and at most in $V_k(S)$ among all sets S of n points, and to better understand the structure of $V_k(S)$, we present two special point sets S, determine subsets $P_k \subset S$ that define a face $f(P_k)$ of $V_k(S)$ and count the number of i-sided faces. For the first point set S, studied in Section 2, all its points are placed very close to the coordinate axes. Among the properties of $V_k(S)$ for this set, we point out that $V_k(S)$ contains no quadrilateral for $3 \leq k \leq \frac{n+1}{4}$, and, for $k \geq \frac{n+3}{2}$, all the bounded faces of $V_k(S)$ are hexagons. The second considered point set S consists of n points on the positive branch of the parabola $y = x^2$, i.e. on the two-dimensional moment curve. We describe all faces of $V_k(S)$ precisely. Interestingly, for every $2 \leq k \leq n-2$, $V_k(S)$ contains exactly one quadrilateral, and for $k \geq 3$, all hexagonal faces are alternating (this is defined in the following). A similiar study of counting the number of i-sided faces in a special point set was carried out for Voronoi diagrams of order 1 in [5], where the points are placed on the Archimedean spiral.

The i-sided faces of $V_k(S)$ can be classified even more precisely: each vertex of a face $f(P_k)$ is either the circumcenter of two points from P_k and one point from $S \backslash P_k$, a type II vertex, or of one point from P_k and two points from $S \backslash P_k$, a type I vertex [3]. Such vertices are also called inner and outer vertices [9], or old and new vertices [7]. It is known that in $V_k(S)$, for $2 \leq k \leq n-2$, every bounded face has vertices of type I and of type II [7]. For $k \geq 2$, every quadrilateral has two vertices of each type, which appear in alternating order. There exist two classes of pentagonal faces: Class I are pentagons with three vertices of type I and two vertices of type II, and Class II are pentagons with three vertices of type II and two vertices of type I. We say that a hexagonal face is *alternating* if its vertices alternate between type I and type II. See [3] for some structural results on alternating hexagons in $V_k(S)$. We then also study the number of faces according to this classification for type I and type

II vertices. We will need the *edge labeling* of $V_k(S)$, defined in [3]. An edge that delimits a face of $V_k(S)$ is a (possibly unbounded) segment of the perpendicular bisector of two points i and j of S. This well-known observation induces a natural labeling of the edges of $V_k(S)$ with the following rules:
Edge rule: An edge of $V_k(S)$ from the perpendicular bisector of points $i, j \in S$ has labels i and j, where label i is on the side (half-plane) of the edge that contains point i and label j is on the other side.
Vertex rule: Let v be a vertex of $V_k(S)$ and let $\{i, j, \ell\} \in S$ be the set of labels of the edges incident to v. The cyclic order of the labels of the edges around v is i, i, j, j, ℓ, ℓ if v is of type I, and it is i, j, ℓ, i, j, ℓ if v is of type II.
Face rule: In each face of $V_k(S)$, the edges that have the same label i are consecutive, and these labels i are either all in the interior of the face, or are all in the exterior of the face.

Using the edge labeling, we prove a structural result that holds for every set of points S, namely that no two quadrilaterals can share an edge in $V_k(S)$, for $k \geq 2$. We also describe the labels of the edges of $V_k(S)$ for the point set on the parabola, studied in Section 3. Proofs are omitted in this abstract.

2 Points close to the axes

Let $S = H \cup V$ where H are all the points of the form $H_i = (i, 0)$ with $i \in \mathbb{Z}, -n \leq i \leq n, n \geq 1$, and V are all of the form $V_j = (0, j)$ with $j \in \mathbb{Z}, -(n+m) \leq j \leq -n, m > 1$, or $n \leq j \leq n+m$. H_n, H_{-n} are called extremes of H, and $V_n, V_{-n}, V_{n+m}, H_{-n-m}$ are extremes of V. We slightly perturb the points of H and V so that the points of S are in general position. The structure of Voronoi diagrams stays the same when the perturbation of the points is sufficiently small; values of k where this perturbation can make a difference are not considered. Note that $|S| = |H| + |V| = (2n+1) + 2m + 2$.

Lemma 1. *Every circle C passing through the points H_i and $H_{i'}$, where $i, i' \in \mathbb{Z}$, encloses all points H_p, with $-n \leq i < p < i' \leq n$. If in addition C passes through V_j, $n \leq j$, then C encloses the points V_ℓ such that $n \leq \ell < j$. Analogously if $j \leq -n$, then C encloses the points V_ℓ such that $j < \ell \leq -n$.*

Lemma 2. *Let C be a circle passing through $H_i \in H$, V_j and $V_{j'} \in H$, $j, j' \in \mathbb{Z}$, where $n \leq j < j'$ or $j' < j \leq -n$. Then, if $i > 0$, C encloses H_p with $i < p$; if $i < 0$, C encloses H_p with $p < i$.*

2.1 Quadrilaterals

Property 3. *$V_1(S)$ has $|H| + |V| - 6 = 2(n+m) - 3$ quadrilateral faces. Also, if the points of S are on the coordinate axes, two edges of each quadrilateral are tangent to the parabolas with focus H_n, H_{-n}, V_n, V_{-n} and directrix an axis.*

To illustrate Property 3, see Figure 1.

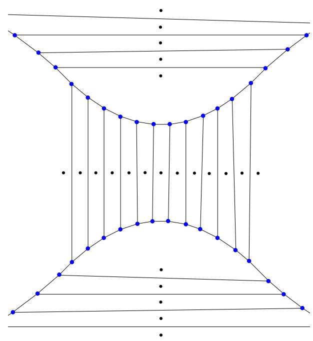

Figure 1: All bounded faces of $V_1(S)$ are quadrilaterals except two of them which have $|H| + 2$ sides.

Property 4. *$V_2(S)$ has four quadrilateral faces: $f(\{V_n, H_n\})$, $f(\{V_n, H_{-n}\})$, $f(\{V_{-n}, H_n\})$ and $f(\{V_{-n}, H_{-n}\})$. Moreover, $V_k(S)$ does not have quadrilateral faces for $3 \leq k \leq |V|/2$ and $k \geq |H| + 2$.*

2.2 Pentagons

It is possible to find a collection of pentagons joined two by two, sharing an edge. We find this configuration in the $V_k(S)$, where $2 \leq k \leq |V|/2 = m+1$ (if $m = n$, then $2 \leq k \leq (|S|+1)/4$). See Figure 2.

Property 5. *In each $V_k(S)$, $2 \leq k \leq |V|/2$, there are two chains of pentagons. Further, if P_k is a set of points associated to a pentagonal face of $V_k(S)$, then P_k has either a single point from V and an extreme point of H, or a single point from H and an extreme point of V, except in the case where $k = 2$, in which the two points of P_2 cannot be one of them extreme of V and the other one extreme of H. The number of pentagonal faces is $2(|V| + |H|) - 12$ in $V_2(S)$ and $2(|V| + |H|) - 4$ in $V_k(S)$, for $k \geq 3$.*

2.3 Hexagons

Property 6. *Let $f(P_k)$ be a non-alternating hexagonal face of $V_k(S)$. Then, P_k is either:*

- *A set of k consecutive points of $H \setminus \{H_{-n}, H_n\}$ where $2 \leq k \leq |H| - 2$.*
- *A set of k consecutive points of $V \setminus \{V_{-n}, V_n, V_{-(n+m)}, V_{n+m}\}$ where $2 \leq k \leq |V|/2 - 2$.*
- *A set of k_1 consecutive points of H that contains either H_{-n} or H_n, together with k_2 consecutive points of $V \setminus \{V_{-(n+m)}, V_{n+m}\}$ that contain either*

V_{-n} or V_n, where $k = k_1 + k_2 \geq 4$, $2 \leq k_1 < |H| - 1$, $2 \leq k_2 < (|V|/2) - 1$.

Property 7. *Let $f(P_k)$ be an alternating hexagonal face of $V_k(S)$. Then P_k is either:*

- *A set of k_1 contiguous points of $H \setminus \{H_{-n}, H_n\}$ together with $k_2 = k - k_1$ contiguous points of $V \setminus \{V_{-(n+m)}, V_{n+m}\}$ that contain V_{-n} or V_n where $k \geq 3$, $2 \leq k_1 < |H| - 2$, $k_2 < (|V|/2) - 1$.*
- *A set of k_2 contiguous points of $V \setminus \{V_{-n}, V_n, V_{-(n+m)}, V_{n+m}\}$ together with $k_1 = k - k_2$ contiguous points of H that contain H_{-n} or H_n, where $2 \leq k_2 < |V|/2 - 2$, $k_1 < |H|$.*

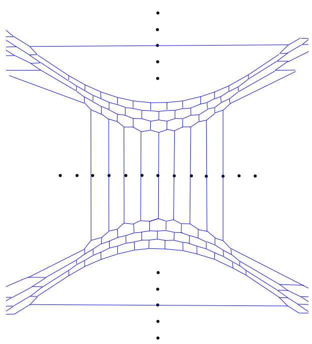

Figure 2: Hexagonal and pentagonal faces in $V_4(S)$.

Property 8. *The numbers of hexagons in $V_k(S)$, for $2 \leq k \leq \min\{|H| - 2, |V|/2 - 2\}$ are:*

$k = 2$	$2n + 2m - 6$
$k = 3$	$6n + 6m - 21$
$k \geq 4$	$(\lvert S\rvert - 3)(2k - 3) - 3k^2 + 4k - 6$

Property 9. *If $|V| \leq |H|$ and $|H| + 2 \leq k < |S| - 1$, then all bounded faces of $V_k(S)$ are hexagons. Moreover, if the set P_k associated to the bounded face $f(P_k)$ of $V_k(S)$ does not contains an extreme point of H, then $f(P_k)$ is an alternating hexagon.*

3 Points on the positive branch of a parabola

Let S be the ordered set of points of the form $Q_i = (x_i, x_i^2)$, where $x_i \in \mathbb{R}$, $x_i > 0$, $i \in \mathbb{N}$, $1 \leq i \leq n$ and $Q_i < Q_j$ if and only if $i < j$ and $x_i < x_j$. We count the bounded faces $V_k(S)$, which can only be a quadrilateral, pentagons and alternating hexagons.

Lemma 10. *Every circle C passing through the points Q_i, Q_j and Q_ℓ, with $i < j < \ell$, encloses all points Q_m with $m < i$ or $j < m < \ell$.*

3.1 Quadrilaterals

Property 11. *$V_k(S)$ with $2 \leq k \leq n - 2$ has a unique quadrilateral face $f(P_k)$. The two labels at the interior of $f(P_k)$ are $k - 1$ and $k + 1$ with $Q_{k-1}, Q_{k+1} \in P_k$ and the two labels at the exterior of $f(P_k)$ are k and $k + 2$ with $Q_k, Q_{k+2} \notin P_k$. Also, $P_k = \{Q_1, Q_2, ..., Q_{k-2}, Q_{k-1}, Q_{k+1}\}$.*

3.2 Pentagons

There exists two classes of pentagonal faces with both types of vertices: Class I are pentagons with three vertices of type I and two vertices of type II and Class II are pentagons with three vertices of type II and two vertices of type I.

Property 12. *Let $f(P_k)$ be a class I pentagonal face of $V_k(S)$ with $2 \leq k \leq n - 2$, and let i and j be the two labels at the interior of $f(P_k)$ with $i < j$ and $Q_i, Q_j \in P_k$. Then, $i = k - 1$, $k + 2 \leq j \leq n - 1$ and the three labels at the exterior of $f(P_k)$ are k, $j - 1$ and $j + 1$, with $Q_k, Q_{j-1}, Q_{j+1} \notin P_k$. Also, $P_k = \{Q_1, Q_2, ..., Q_{k-2}, Q_{k-1}, Q_j\}$.*

Property 13. *Let $f(P_k)$ be a class II pentagonal face of $V_k(S)$ with $3 \leq k \leq n - 3$, and let i, j, ℓ be the three labels at the interior of $f(P_k)$ with $i < j < \ell$ and $Q_i, Q_j, Q_\ell \in P_k$. Then, $1 \leq i \leq k - 2$, $j = i + 2$, $\ell = k + 1$ and the three labels at the exterior of $f(P_k)$ are $i + 1$ and $k + 2$, with $Q_{i+1}, Q_{k+2} \notin P_k$. Also, the points Q_m with $m < i$ or $i + 2 < m < k + 1$ are the remaining points of P_k.*

Property 14. *$V_k(S)$ with $2 \leq k \leq n - 2$, has exactly $(n - k - 2)$ class I pentagonal faces.*

Property 15. *$V_k(S)$ with $3 \leq k \leq n - 3$, has exactly $(k - 2)$ class II pentagonal faces.*

3.3 Hexagons

Property 16. *Let $f(P_k)$ be an alternating hexagonal face of $V_k(S)$ with $3 \leq k \leq n - 3$, and let i, j and ℓ be the three labels at the interior of $f(P_k)$ with $i < j < \ell$ and $Q_i, Q_j, Q_\ell \in P_k$. Then, $1 \leq i \leq k - 2$, $i + 2 \leq j \leq n - k + i$, $\ell \leq n - 1$ and the three labels at the exterior of $f(P_k)$ are $i + 1$, $j - 1$ and $\ell + 1$ with $Q_{i+1}, Q_{j-1}, Q_{\ell+1} \notin P_k$. Also, the points Q_m with $m < i$ or $j < m < \ell$ are the remaining points of P_k.*

Property 17. *$V_k(S)$ with $3 \leq k \leq n - 3$, has exactly $(k - 2)(n - k - 2)$ alternating hexagons.*

4 Experimental and theoretical results

Previous properties have been additionally verified computationally. For this, a generator algorithm for the order-k Voronoi diagram was implemented in Python, so n-sided bounded faces can be counted. This code was used to seek for more general properties. We generated 1000 sets of n uniformly distributed random points on the unit square in general position for each n from 4 to 20. We obtained all order-k Voronoi diagrams for these sets. Then, minimum, maximum and mean of the n-sided bounded faces for all of the same order Voronoi diagrams for the sets with the same number of points were computed. We get tables like the ones below shown for $n = 10$.

$n = 10$	Quadrilateral	Pentagons	Hexagons
$k = 1$	min= 0 max= 5 mean= 1.379	min= 0 max= 5 mean= 1.464	min= 0 max= 3 mean= 0.684
$k = 2$	min= 0 max= 6 mean= 3.077	min= 0 max= 12 mean= 4.177	min= 0 max= 10 mean= 2.988
$k = 3$	min= 0 max= 8 mean= 3.957	min= 0 max= 12 mean= 4.644	min= 0 max= 11 mean= 4.16
$k = 4$	min= 0 max= 9 mean= 4.092	min= 0 max= 11 mean= 4.719	min= 0 max= 13 mean= 4.342
$k = 5$	min= 0 max= 10 mean= 3.726	min= 0 max= 12 mean= 4.249	min= 0 max= 11 mean= 3.908
$k = 6$	min= 0 max= 7 mean= 3.007	min= 0 max= 10 mean= 3.432	min= 0 max= 9 mean= 3.017
$k = 7$	min= 0 max= 5 mean= 2.048	min= 0 max= 8 mean= 2.328	min= 0 max= 7 mean= 1.866
$k = 8$	min= 0 max= 3 mean= 0.978	min= 0 max= 5 mean= 1.190	min= 0 max= 3 mean= 0.615

$n = 10$	Class I Pentagons	Class II Pentagons	Alternating Hexagons
$k = 1$	min= 0 max= 0 mean= 0	min= 0 max= 0 mean= 0	min= 0 max= 0 mean= 0
$k = 2$	min= 0 max= 12 mean= 4.177	min= 0 max= 0 mean= 0	min= 0 max= 0 mean= 0
$k = 3$	min= 0 max= 9 mean= 3.335	min= 0 max= 6 mean= 1.309	min= 0 max= 7 mean= 1.081
$k = 4$	min= 0 max= 7 mean= 2.678	min= 0 max= 7 mean= 2.678	min= 0 max= 8 mean= 1.304
$k = 5$	min= 0 max= 6 mean= 1.983	min= 0 max= 7 mean= 2.266	min= 0 max= 7 mean= 1.304
$k = 6$	min= 0 max= 5 mean= 1.241	min= 0 max= 7 mean= 2.190	min= 0 max= 7 mean= 1.216
$k = 7$	min= 0 max= 3 mean= 0.534	min= 0 max= 6 mean= 1.793	min= 0 max= 4 mean= 0.489
$k = 8$	min= 0 max= 0 mean= 0	min= 0 max= 5 mean= 1.190	min= 0 max= 0 mean= 0

Note that, since for $k = n - 1$ the Voronoi diagram $V_k(S)$ has no bounded faces, there is no row in the tables for $k = 9$ as all the values are always 0.

With these tables we try to find general properties for the number of quadrilaterals, pentagons, and hexagons in higher order Voronoi diagrams. We proved the next results for the bounded faces of the Voronoi diagrams of any set of points in general position.

Property 18. *Only in Voronoi diagrams of order one, it is possible to find two quadrilaterals sharing an edge.*

Property 19. *$V_k(S)$ with $k \geq 2$, cannot have a bounded face with only two type II vertices and sharing a type I vertex with two Class II pentagonal faces.*

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