Minsum m watchmen's routes in Stiegl polygons*†

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1 Introduction

In the classical watchman route problem, we aim for the shortest closed route R within a polygon P, such that all points of P are visible to some point of R.

Carlsson *et al.* [1] introduced the m-watchmen problem as a generalization of this problem: instead of a single mobile guard, we are given m mobile watchmen (with or without a given starting point) and we aim to find routes for all watchmen, such that all points in P are visible from at least one route and such that the sum of the watchman-route lengths is minimized. Carlsson et al. showed that the problem is NP-hard in simple polygons and provided a polytime algorithm in histograms. Nilsson and Wood [2, 3] gave an $O(n^2m)$ time and $O(n^2)$ storage algorithm for spiral polygons without given starting points for the m watchmen. Nilsson and Schuierer [4] also considered histograms, but altered the objective to minimizing the length of the longest of the m watchmen routes, for which they provided an $O(n^2 \log n)$ algorithm. Also, Mitchell and Wynters [5] considered the minmax criterion. They gave an $O(n^4m)$ algorithm for rectilinear vision in rectilinear monotone polygons and showed that the problem is NP-hard for $m = 2$ in general. Nilsson and Packer [6] gave an approximation algorithm for two watchmen in simple polygons. Packer [7] presented heuristics for both the minmax and the minsum criterion in polygons with and without holes. In this paper, we present an $O(n^2 \cdot \min\{m, n\})$ time and $O(n \cdot \min\{m, n\})$ storage algorithm to compute the minsum set of m watchmen routes given their starting points in a Stiegl polygon—which we define in Section 2. Without starting points the solution is trivially a single point.

2 Watchmen routes' properties in Stiegl polygons

A staircase polygon P, as defined in [8], is called a *Stiegl polygon* if the floor solely consists of one horizontal and one vertical edge, which we call the *base* and the *wall* of P, respectively. Moreover, we call the vertex between the base and the wall the *origin* of P.

Let S be a set of m points in the interior of P which we consider as starting positions for the watchmen. We say that point p sees point q if the segment $[p, q]$ lies in P. In particular, it can partly be on the boundary of P, hence, one can see along a boundary edge of P. We denote the x-coordinate of a point p by $x(p)$, and the y-coordinate by $y(p)$. We furthermore denote the horizontal and the vertical segment that goes through point p and lies inside P with $h(p)$, and $v(p)$, respectively.

Definition 1 *For two points p and q, if $x(p) \geq x(q)$ and $y(p) \leq y(q)$, we say that p* dominates *q.*

Observe that, if p dominates q its visibility polygon is a superset of the visibility polygon of q and that any watchman can be limited to walk to the right and downwards from its starting point, because the bottom-rightmost position on its route w dominates all other positions on w. Hence:

Lemma 2 *A watchman route having starting point s and rightmost x-coordinate x, and lowest y-coordinate y can be replaced by the segment $[s, (x, y)]$ without increasing the minmax or minsum value of the solution.*

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*Any connection to an Austrian beer manufacturer is purely coincidental.

†A.Br., B.J.N, and C.S. are supported by the Swedish Research Council project "ILLUMINATE provably good methods for guarding problems" (2021-03810). B.J.N. and C.S. are supported by the Swedish Research Council project "New paradigms for autonomous unmanned air traffic management" (2018-04001).

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Because all watchmen routes are a segment that the watchman walks back and forth, for the rest of this paper, we only compute that length of the segment and the routes have twice the length we compute.

Lemma 3 *If s and s' are starting points such that s dominates s', and w, w' are two watchmen routes starting at s and s', respectively, then w' will have zero length in an optimal minsum solution.*

Proof. Everything s' sees is seen by s. If a point p is unseen both from s and s', but the watchman starting at s' sees it, then the watchman has to cross either the horizontal or vertical line through s. W.l.o.g., the horizontal one, $h(s)$. But then the distance from s to a point that sees p is shorter than that from s'. ☐

Hence, let S be the set of non-dominated starting points. S admits a total order, so let the points be sorted from bottom-left to top-right: $s_1 < \cdots < s_m$. Define the x-overlap of two watchmen w and w' as the intersection between the projection of w and w' onto the base. Similarly, define the y-overlap as the intersection of the projections onto the wall.

Lemma 4 *Let W be a set of optimal watchmen routes and let $w, w' \in W$ be two watchmen routes with starting points s and s', respectively. If neither w nor w' has length zero, then w and w' have no x- and no y-overlap.*

Proof. Assume w.l.o.g. $s < s'$ and that the x-overlap of w and w' is non-empty. Let w and w' be disjoint (otherwise, we can shorten the routes). Let p be w's endpoint, and p' be (w')'s endpoint. Observe that $x(p) < x(p')$ as otherwise p would dominate the route w'. Let the overlap be the interval $[x_1, x_2]$, then $x(s') = x_1$. We can shorten w: Substitute w by $[s, (x(s'), y(p))]$. The vertical segment $[s', (x(s'), y(p))]$ is fully contained in P. Hence, no convex corner that w saw before is unseen by the new w and w'. By symmetry, it also holds for the y-overlap. ☐

Let P be a Stiegl polygon, and let C be the convex corners on the ceiling. Enumerate the corners in C from bottom-left to top-right by $c_1, \ldots, c_{\tilde{n}}$, where $\tilde{n} = \frac{n-2}{2}$. For a convex corner c_k let $h_k = h(c_k)$ be the extension of the horizontal edge at c_k, and $v_k = v(c_k)$ the extension of the vertical edge at c_k.

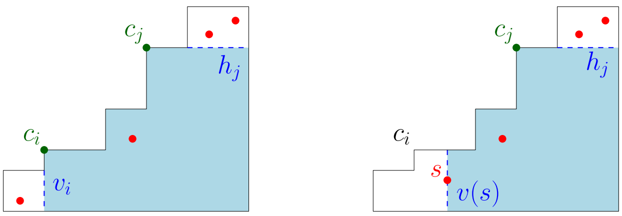
Lemma 5 *Let W be a set of optimal watchmen routes. Then, for every convex corner c_k that is not seen from S, either extension h_k or v_k is visited, no such extension is visited twice, and every watchman stops at an extension.*

Proof. First, we show that every watchman stops at an extension. Assume w.l.o.g. that watchman w crosses extension v_k in $v_k^\times$, and that this is the last extension on its route. Let q be the last point on its route. When walking along segment $[v_k^\times, q]$, w will not see any yet unseen convex corner that he did not see at $v_k^\times$. Hence, w can be replaced by $[s, v_k^\times]$, contradicting the assumption that it was optimal. Next, we argue that for each c_k that is not seen from S, either h_k or v_k is visited. Assume w.l.o.g. that watchman w visits v_k and stops there. Then, he can see all of the rectangle between c_k and the origin, but he will not see c_{k+1}. Let c_{k+1} be guarded by watchman w'. If w' moves to extension h_k, he will not see anything that w does not see yet. In order to see some convex corner c_j, $j < k$, that is not seen from w, w' has to walk downwards to a point below $v_k^\times$, but then w and w' have non-empty y-overlap, contradicting Lemma 4. Finally, no extension is visited twice since this would mean that two watchmen have nonempty x- or y-overlap, again contradicting Lemma 4. ☐

Let W be a set of watchmen that guard P optimally. Then the watchmen can be separated into solutions of subpolygons with bottom and right edges given by the extensions that are visited by the watchmen, and where the subpolygons are separated by *crates*, which are solely guarded by the starting points inside, and the watchmen outside, but no watchman moves inside these crates. A *crate* is a Stiegl polygon with precisely two convex corners on the ceiling that are not seen by the set of starting points in the crate. Specifically we define a crate by (a) two unseen convex corners c_i and c_j where every $c_k, i < k < j$, is seen, where we cut along the extensions v_i, h_j, or (b) one unseen convex vertex c_j and a starting point s where we cut along $v(s)$ and h_j, or v_j and $h(s)$ depending on the position of s, and s is considered to be outside the crate. In case there are two starting points s, s' with $x(c_i) < x(s) < x(s') < x(c_{i+1})$, we only consider the crate cut at $v(s)$, but not the crate cut at $v(s')$. Similarly, for two starting points s, s' with $y(c_{j-1}) < y(s) < y(s') < y(c_j)$, we only consider the crate cut at $h(s')$. We say such a crate starts at i and ends at j. (We do not define a crate if both cuts pass starting points as then the cuts are automatically visited.) The two unseen convex corners on the crate's ceiling are precisely those incident to these cuts. See Figure 1 for the different types of crates.

3 A dynamic programming algorithm

We describe an algorithm, which iteratively splits the polygon into two independent subpolygons, called *sub-Stiegl polygons*, that are separated by a crate and which computes the minimum length watchmen routes in each of them recursively. In each recursion,



(a) Crate cut at v_i and h_j

(b) Crate cut at $v(s)$ and h_j

Figure 1: The two different types of crates.

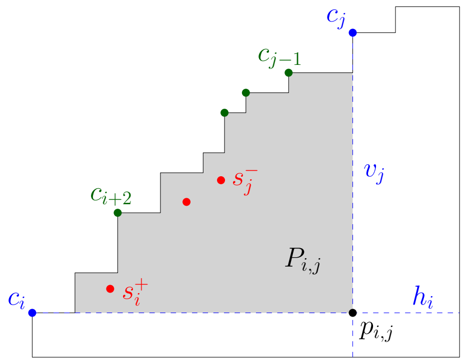


Figure 2: The sub-Stiegl polygon $P_{i,j}$

it is ensured that the neighboring crates are seen by forcing watchmen to walk to the base and the wall of the sub-Stiegl polygon.

3.1 Idea

After cutting out a crate, we are left with two sub-Stiegl polygons, which need to be guarded. For the lower sub-Stiegl polygon, we consider the minimum length of a watchman route guarding it immediately, and all possible crates that split the subpolygon. The upper one will be guarded immediately.

Since it is not necessary to walk to the bottom and the right boundary of the initial polygon P, some pre-processing is necessary. We do this by cutting off a horizontal strip along the lower boundary of P, and a vertical strip along the right boundary of P such that the interior of the strip is seen by a watchman that visits the extension along which we cut.

For defining the horizontal strip, consider the first convex corner c_1. If no starting point lies below h_1, this is an extension which needs to be visited in order to see c_1. Therefore we cut off the horizontal strip below h_1. If there is a starting point s below h_1, we consider $h(s)$ as the first extension, and cut off the horizontal strip below. Note that it does not matter whether we cut off the strip below the lowermost starting point or any other starting point as long as we do not cut off an unseen convex corner on the ceiling, because the extension will be visited from the starting point we choose with a watchman of length 0, and the shortest watchman route to a vertical extension will never start at any of the lower starting points as they also lie further to the left. Analogously, we cut off a vertical strip at the wall of P.

3.2 Sub-Stiegl polygons

In each iteration, the algorithm considers a sub-polygon of P and computes the optimal solution within that.

Let $P_{i,j}$ be the Stiegl polygon that evolves when cutting off a crate along h_i, or along $h(s)$ for any $s \in S$ satisfying $y(c_{i-1}) < y(s) \leq y(c_i)$, and v_j, or $v(s')$ for any $s' \in S$ satisfying $x(c_j) < x(s') \leq x(c_{j+1})$. This definition is unique up to the choice of the starting point that defines the cut. Here, we simply choose the leftmost possible starting point for vertical cuts, and the uppermost possible starting point for horizontal cuts in order to remove a maximal crate. This will not change the solution in $P_{i,j}$ because among all possible starting points s satisfying $y(c_{i-1}) < y(s) \leq y(c_i)$, the uppermost one has the shortest direct path to the wall of $P_{i,j}$ among all starting points below h_i, and the same holds for any possible vertical cut.

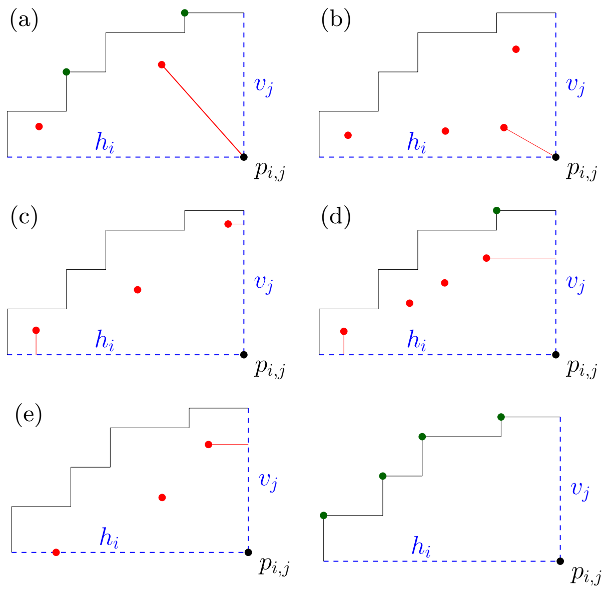
Let $p_{i,j}$ be the origin of $P_{i,j}$, let $S_{i,j}$ be the subset of starting points in S that lie in $P_{i,j}$ (possibly on the boundary) and let $s_i^+ < \cdots < s_j^-$ be the points in $S_{i,j}$. Let furthermore $C_{i,j}$ be the subset of C that lies in $P_{i,j}$ and is not yet seen. See Figure 2 for an illustration. The goal is to visit both the floor and the wall of $P_{i,j}$ with watchmen routes that start at $S_{i,j}$, such that all corners in $C_{i,j}$ are seen.

Let $\mathcal{L}(i)$ be the length of the minimum watchmen routes in the subpolygon $P_{1,i}$, starting at the points $S_{1,i}$.

Lemma 6 *If every convex corner in a sub-Stiegl polygon $P_{i,j}$ is already seen by $S_{i,j}$, then the optimal watchmen routes inside $P_{i,j}$ consists either of one watchman starting at $s \in S_{i,j}$ who directly moves to $p_{i,j}$, or s_i^+ who moves vertically down to h_i and s_j^- who moves horizontally right to v_j.*

Proof. As all convex corners are already seen, there is no extension inside $P_{i,j}$ that needs to be visited by a watchman. Hence, any watchman will directly walk to h_i or v_j and stop there. Moreover, if a watchman walks towards only one of the extensions, but does not visit the other one, its shortest route will be the orthogonal onto the extension. For any such watchman route starting neither at s_i^+ nor at s_j^-, its route can be replaced by the parallel route starting at s_i^+ or s_j^-, respectively. □

We define the *uninorm* of a polygon $P_{i,j}$, denoted $\|P_{i,j}\|_u$, as the length of the shortest possible watchmen routes from which $P_{i,j}$ is guarded, using starting



(f) $S_{i,j} = \varnothing, C_{i,j} \neq \varnothing$
$\Rightarrow \|P_{i,j}\|_u = \infty$

Figure 3: Different solutions for sub-Stiegl polygon $P_{i,j}$, depending on the starting points $S_{i,j}$ (red points). The floor and the right wall (blue dashed lines) need to be visited and all unseen convex corners $C_{i,j}$ (green) need to be guarded.

points in $S_{i,j}$, and that visit both the base and the wall of $P_{i,j}$, and such that the solution is not split into a set of independent solutions. The value of the uninorm depends on the unseen convex corners and the starting points in $S_{i,j}$ (the precise dependency is given by the $(*)$-condition that we define in the last paragraph of this section),

$$\|P_{i,j}\|_u = \begin{cases} \min\limits_{s \in S_{i,j}} \|s, p_{i,j}\| \quad \text{if } C_{i,j} \neq \varnothing, |S_{i,j}| \geq 1 \text{or } \neg(*) \\ \min\left\{ \|s_i^+, h_i\| + \|s_j^-, v_j\|, \min\limits_{s \in S_{i,j}} \|s, p_{i,j}\| \right\} \\ \qquad\qquad\qquad\qquad \text{if } |S_{i,j}| \geq 1 \text{ or } (*) \\ \infty \quad \text{if } C_{i,j} \neq \varnothing, S_{i,j} = \varnothing \text{ or } P_{i,j} \text{ degenerates.} \end{cases}$$

In case $S_{i,j} = \varnothing$ while $C_{i,j} \neq \varnothing$, then $P_{i,j}$ cannot be guarded from its interior. Hence, $\|P_{i,j}\|_u$ is defined to be ∞. See Figure 3f. If $P_{i,j}$ is a degenerate crate with no area, then again $\|P_{i,j}\|_u = \infty$.

If every unseen convex corner $c \in C_{i,j}$ satisfies $y(c) < y(s_i^+)$ or $x(c) > x(s_j^-)$ then we say that $P_{i,j}$ satisfies the $(*)$-condition. If $P_{i,j}$ satisfies $(*)$, a watchman starting at s_i^+ who moves vertically down to h_i and a second watchman starting at s_j^- who moves horizontally right to v_j is a candidate solution (see Figure 3c–3e). The other candidate solutions are given by a single watchman moving from a starting point in $S_{i,j}$ to the origin $p_{i,j}$ (see Figure 3a–3b). The uninorm is then the minimum over all candidate solutions.

3.3 The algorithm

The total length of the minimum watchmen routes is computed by the recursion

$$\mathcal{L}(j) = \min \begin{cases} \|P_{1,j}\|_u \quad \text{or} \\ \min\limits_{\substack{1<i<j-1 \\ i \text{ unseen or} \\ \exists s: x(c_{i-1})<x(s)<x(c_i)}} \left\{ \mathcal{L}(i) + \min\limits_{\substack{i<k<j \\ i,k \text{ define a crate}}} \|P_{k,j}\|_u \right\}, \end{cases}$$

where the current Stiegl polygon is either guarded immediately, using watchmen routes of length $\|P_{1,j}\|_u$, or split into two sub-Stiegl polygons where the upper one, $P_{k,j}$, is guarded immediately. We precompute the uninorm of all sub-Stiegl polygons in $O(n(n+m)\log^2 m)$ time (per subpolygon $P_{i,j}$, query the closest point to the origin in $O(\log^2 m)$ time using a dynamic closest point data structure [9]). To fill out the lookup-table position $\mathcal{L}(j)$, the dynamic programming algorithm considers all values $\mathcal{L}(i)$, $i < j$, and corresponding values $\|P_{k,j}\|_u$ with index $k > i$, where i and k define a crate, and computes their sum. There are less than j values for the start of the crate i, and at most $j - i$ ends of the crate that need to be verified since for every convex corner, we only consider the maximum possible crate. As every lookup takes constant time, we can compute each entry in $O(n \cdot \min\{m, n\})$ time. Hence, the algorithm takes $O(n^2 \cdot \min\{m, n\})$ time and $O(n \cdot \min\{m, n\})$ storage.

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