Characterizing rotation systems of generalized twisted drawings via 5-tuples

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Simple drawings are drawings of graphs in which the edges are Jordan arcs and each pair of edges share at most one point (a proper crossing or a common endpoint). The rotation of a vertex is the cyclic order of its incident edges. The rotation system of a simple drawing is the collection of the rotations of all vertices. An abstract rotation system of K_n gives for every vertex v, a cyclic order of the other $n-1$ vertices. Not every abstract rotation system can be realized as a simple drawing, but this realizability can be checked in $O(n^5)$ time [1]. Two simple drawings of K_n have the same crossing edge pairs if and only if they have the same or inverse rotation systems [2]; such drawings are called weakly isomorphic.

A simple drawing D of K_n is generalized twisted if there exists a point O such that each ray emanating from O crosses each edge of D at most once and there exists a ray r emanating from O that crosses all edges of D. We call a rotation system generalized twisted if there exists a generalized twisted drawing with that rotation system. Generalized twisted drawings have been used to improve bounds on plane substructures in (general) simple drawings of complete graphs [3]. Moreover, they are the biggest class for which it is known that each drawing has exactly $2n-4$ empty triangles [4], which is conjectured to be the minimum for all simple drawings. In addition to possibly being useful for further general results, generalized twisted drawings are quite interesting in their own right.

In this talk, we present the following new characterization of generalized twisted drawings via abstract rotation systems.

Theorem 1 *Let R be an abstract rotation system of K_n with $n \geq 7$. Then R is generalized twisted if and only if every rotation sub-system induced by 5 vertices is generalized twisted.*

Up to relabeling, there is one unique rotation system of K_5 that is generalized twisted[1] , and one unique rotation system R of K_6 such that every rotation subsystem of R induced by 5 vertices is generalized twisted, but R is not generalized twisted [3]. In particular, Theorem 1 does not hold for $n = 6$.

To prove Theorem 1, we computationally verify it for $7 \leq n \leq 10$ and then use this as an induction base to prove the statement for general n. We use the following two concepts as our main ingredients. (1) A pair of cells in a drawing D is an *antipodal vi-cell pair*, if both cells have a vertex on their boundary and for any triangle of D, the two cells lie on different sides. A simple drawing D of K_n is weakly isomorphic to a generalized twisted drawing if and only if D has two antipodal vi-cells [3]. We show that any simple drawing D has at most two antipodal vi-cell pairs, and it has exactly two if and only if there is an edge e of D such that e crosses every edge in D not adjacent to e. (2) A vertex-empty triangle xyz is an *empty star triangle at* x if no edge incident to x crosses yz. For every vertex x of a generalized twisted drawing D, there are two empty star triangles at x [4]. We show that D has exactly two pairs of antipodal vi-cells if and only if there is a vertex v such that the two empty star triangles at v are adjacent.

References

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[1]The unique rotation system of K_5 that is 'twisted'.