Recognizing rotation systems of generalized twisted drawings in $O(n^2)$ time

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Simple drawings are drawings of graphs in which the edges are Jordan arcs and each pair of edges share at most one point (a proper crossing or a common endpoint). The rotation of a vertex is the cyclic order of its incident edges. The rotation system of a simple drawing is the collection of the rotations of all vertices. Two simple drawings of K_n have the same crossing edge pairs if and only if they have the same or inverse rotation systems [1].

A simple drawing D of K_n is generalized twisted if there exists a point O such that each ray emanating from O crosses each edge of D at most once and there exists a ray r emanating from O such that all edges of D cross r. We call a rotation system generalized twisted if there exists a generalized twisted drawing with that rotation system. Generalized twisted drawings have been used to improve bounds on plane substructures in (general) simple drawings of complete graphs [2]. Moreover, they are the biggest class for which it is known that each drawing has exactly $2n-4$ empty triangles [3], which is conjectured to be the minimum for all simple drawings. In addition to being useful for proving general results, generalized twisted drawings are quite interesting in their own right.

In this talk, we present an efficient algorithm to decide if a rotation system is generalized twisted.

Theorem 1 *Let R be the rotation system of a simple drawing of K_n. Then deciding whether R is generalized twisted can be done in $O(n^2)$ time.*

To obtain the algorithm, we first show the following statement for generalized twisted drawings; see Figure 1 for an illustration.

Theorem 2 *Let R be the rotation system of a simple drawing D of K_n and let V be the set of vertices of D.*

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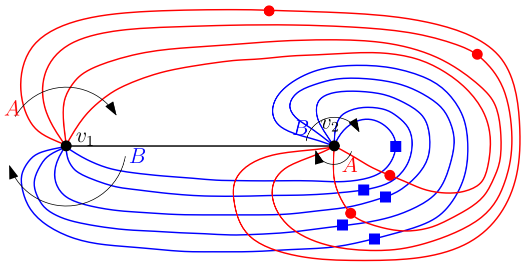

Figure 1: The vertices v_1 and v_2 are as in Theorem 2; arrows around them and colors indicate the partition classes. Only edges incident to v_1 or v_2 are depicted.

Then R is generalized twisted if and only if there exist two vertices v_1 and v_2 in V and a bipartition $A \cup B$ of the vertices in $V \setminus \{v_1, v_2\}$, where some of A or B can be empty, such that: 1. For every pair of vertices u_1 and u_2 that are either both in A or both in B, the edge (u_1, u_2) crosses the edge (v_1, v_2). 2. For every pair of vertices $a \in A$ and $b \in B$, the edge (a, b) does not cross (v_1, v_2). 3. Beginning at v_2 (respectively v_1), in the rotation at v_1 (respectively v_2), all the vertices in B appear before all the vertices in A.

The algorithm then runs in two steps. In the first step, we find $O(1)$ possible candidates for v_1 and v_2 as in Theorem 2, using properties on empty triangles in generalized twisted drawings from [3]. In the second step, we check whether one of those candidates fulfills the requirements of Theorem 2. We show that each of the two steps can be done in $O(n^2)$ time.

References

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