Crossing minimal and generalized convex drawings: 2 non-hard problems

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Simple drawings are drawings of graphs in the plane such that each pair of edges meets in at most one point, either a common endvertex or a crossing. In this work we study two problems on simple drawings that are hard in general but get easy on a certain subclass. As the first problem, Arroyo et al. [1] showed that it is NP-complete to decide whether a specific edge can be added to a simple drawing of a non-complete graph without violating simplicity. In contrast to this, by Levi's Extension Lemma, every pseudolinear drawing can be extended by any set of edges. We show a similar result for *crossing minimal drawings*, that is, drawings of a graph G which contain the minimum number of crossings over all drawings of G.

Theorem 1 *Let $\mathcal{D}$ be a crossing minimal drawing of a graph on n vertices. Then $\mathcal{D}$ can be extended to a simple drawing of the complete graph K_n.*

Proof idea. Note that every crossing minimal drawing $\mathcal{D}$ is a simple drawing. In a first step we show that adding a single edge such that it creates a minimum number of additional crossings results in a simple drawing $\mathcal{D}'$. However, $\mathcal{D}'$ need not be crossing minimal anymore. So in a second step we add a set of edges simultaneously to $\mathcal{D}$ such that each single added edge has a minimum number of crossings with $\mathcal{D}$. Over all possibilities to do so, we then show that choosing a drawing $\mathcal{D}''$ which in addition minimizes the total number of crossings ensures that $\mathcal{D}''$ is simple. □

While it is known that no crossing minimal drawing of K_n is pseudolinear for large enough n, Arroyo et al. [2] asked the question whether all crossing minimal drawings of K_n might be *generalized convex drawings* (short *g-convex*). These are simple drawings where every triangle has a *convex side* Δ, that is, for each pair of vertices in Δ also the edge connecting them lies completely inside Δ. If there exists a choice of a convex side for each triangle such that every triangle T_2, being contained in the convex side Δ_1 of a triangle T_1, has its convex side Δ_2 contained in Δ_1, then the drawing is called *hereditarily convex* (short *h-convex*).

This brings us to the second problem. García et al. [3] showed that it is NP-complete to decide whether a simple drawing $\mathcal{D}$ of K_n contains a plane (no two edges cross) subdrawing with a given number of edges. In this context we call a subdrawing of $\mathcal{D}$ *maximal plane* if it is plane and no edge of $\mathcal{D}$ can be added to it without violating planarity. We call a subdrawing *maximum plane* if it is plane and contains the highest number of edges over all plane subdrawings of $\mathcal{D}$. If a plane subdrawing contains $3n - 6$ edges, then we call it a *combinatorial triangulation*.

Theorem 2 *Let $\mathcal{D}$ be a g-convex drawing of K_n. Then every maximal plane subdrawing of $\mathcal{D}$ is maximum plane. Moreover, if $\mathcal{D}$ is h-convex but not pseudolinear, then every maximal plane subdrawing of $\mathcal{D}$ is a combinatorial triangulation.*

We can show this by combining some results from [2] and [3]. We can further confirm by computer that all crossing minimal drawings of K_n for $n \leq 12$ are h-convex. Since all crossing minimal straight-line or pseudolinear drawings are known to have a triangular convex hull, this gives rise to the following conjecture.

Conjecture 3 *Let $\mathcal{D}$ be a crossing minimal drawing of K_n for $n \geq 3$. Then every maximal plane subdrawing of $\mathcal{D}$ is a combinatorial triangulation.*

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References

[1] Alan Arroyo, Fabian Klute, Irene Parada, Raimund Seidel, Birgit Vogtenhuber, and Tilo Wiedera. Inserting one edge into a simple drawing is hard. *Discrete & Computational Geometry*, 69(3):745–770, 2023. doi:10.1007/s00454-022-00394-9.

[2] Alan Arroyo, Dan McQuillan, R. Bruce Richter, and Gelasio Salazar. Convex drawings of the complete graph: topology meets geometry. *Ars Mathematica Contemporanea*, 22(3):27, 2022. doi:10.26493/1855-3974.2134.ac9.

[3] Alfredo García, Alexander Pilz, and Javier Tejel. On plane subgraphs of complete topological drawings. *Ars Mathematica Contemporanea*, 20(1):69–87, 2021. doi:10.26493/1855-3974.2226.e93.

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