The Borsuk number of geometric graphs

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In 1933, Karol Borsuk wondered if every set X in $\mathbb{R}^d$ could be partitioned into $d+1$ closed (sub)sets each with diameter smaller than that of X [1]. Here, the diameter is defined as the maximum of the distances between two points in the set, under the Euclidean metric. This leads to the concept of the *Borsuk number*. For a set $X \subset \mathbb{R}^d$, the Borsuk number $b(X)$ is the smallest number such that X can be partitioned into $b(X)$ subsets, each with diameter smaller than that of X. The answer to Borsuk's question was shown to be positive for $d = 2, 3$, and for general d for centrally symmetric convex bodies and smooth convex bodies. To the surprise of many researchers, the general answer turned out to be negative, as shown in 1993 by Kahn and Kalai [2]. Since then, research on variants of the Borsuk problem has continued in a plethora of directions, see [3] for a recent survey.

In this work, we propose a formulation of the problem in the context of geometric graphs. A *(plane) geometric graph* is a plane undirected graph $G = (V, E)$ whose vertices are points in $\mathbb{R}^2$, and whose edges are straight-line segments connecting pairs of points. In addition, each edge has a weight equal to the Euclidean distance between its endpoints. We are interested in the *locus* of G, denoted by $\mathcal{L}_G$, which is the set of all points of the plane that are on G. Thus, we treat both G and $\mathcal{L}_G$, interchangeably, as a closed point set. The distance between two points in $\mathcal{L}_G$ is the length of a shortest path between them in G (note such a path will contain up to two fragments of edges, if the points are not vertices). The *diameter* of $\mathcal{L}_G$ or *(continuous) diameter* of G is the maximum distance between any two points in $\mathcal{L}_G$. In contrast to (abstract) graphs, in a geometric graph, there can be an infinite number of pairs of points whose distance is equal to the diameter.

We extend the concept of Borsuk number to geometric graphs. Conceptually, it is the smallest number $b(G)$ such that G can be partitioned into $b(G)$ subgraphs, each with smaller diameter than $\mathcal{L}_G$. However, we need to define carefully how a geometric graph can be partitioned. We consider partitions of $\mathcal{L}_G$ by a sequence of cuts with straight lines. A line ℓ naturally partitions $\mathcal{L}_G$ into two geometric subgraphs (possibly, one empty). Moreover, to guarantee that the partition by ℓ does not produce a disconnected subgraph, we add to both subgraphs the maximal segment of ℓ intersecting $\mathcal{L}_G$.[1] Figure 1 (left) illustrates this for a square. After partitioning the square with a vertical line ℓ (dashed) through its center point, we obtain two subgraphs: all points of $\mathcal{L}_G$ on each halfplane induced by ℓ, union the longest segment in ℓ with endpoints in $\mathcal{L}_G \cap \ell$. Since this partitions the graph into two subgraphs (of $\mathcal{L}_G \cup \ell$), each with smaller diameter than that of $\mathcal{L}_G$, its Borsuk number is two (best possible). However, sometimes more subgraphs are needed. The example in Figure 1 (right) shows a 4-star graph, requiring at least two lines, giving at least three subgraphs. Thus its Borsuk number is three.

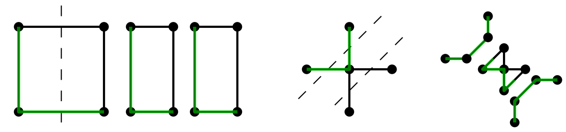

Figure 1: Left: a square with side length 1 and diameter 2 (given by green paths), and a partition with a line. Right: a 4-star partitioned into three subgraphs.

This illustrates the main question studied in this work: What is the Borsuk number of a geometric graph? Clearly, the answer depends on the graph.

In this talk, we will show that, in general, any geometric graph with n vertices can be partitioned with lines into $2n$ subgraphs with smaller diameter. Moreover, we will give upper bounds for $b(G)$ that depend on the number of disjoint diameter paths in the graph. We will also show that the Borsuk number of a tree can be two or three, and discuss how to efficiently figure it out. Finally, we will mention several open problems in relation to this new concept.

References

[1] K. Borsuk. Three theorems on the n-dimensional sphere. *Fund. Math*, 20:177–190, 1933.

[2] J. Kahn and G. Kalai. A counterexample to borsuk's conjecture. *B. Am. Math. Soc.*, 29(1):60–62, 1993.

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[1]So, actually, the partition gives two subgraphs of $\mathcal{L}_G \cup \ell$.