On the number of drawings of a combinatorial triangulation

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In 1962, Tutte [5] proved that the number of triangulations, that is maximal planar graphs with a fixed face with vertices a, b, and c, and n additional vertices is $\psi_{n,0} = \frac{2}{n(n+1)}\binom{4n+1}{n-1} = \Theta\left(\frac{1}{n^{5/2}} 9,\overline{481}^n\right)$. See [5] for a precise definition. We call these triangulations *combinatorial* triangulations. Note that in a combinatorial triangulation, the edges need not be straight-line segments. In contrast to combinatorial triangulations, there is no general formula for the number of *geometric* triangulations, which are defined for a given set S of n points in the plane. A geometric triangulation on S is a maximal planar straight-line graph with vertices the set S. Finding the maximum number $tr(n)$ of geometric triangulations, among all sets S of n points in general position in the plane, is a longstanding open problem in Discrete Geometry. The current best bounds are $\Omega(9,08^n) \leq tr(n) \leq O(30^n)$, [2, 3]. In [4] the question was raised if the numbers of combinatorial and geometric triangulations are somehow related? See Fig. 1 for an example that shows the three combinatorial triangulations on five vertices, but only two of them are geometric triangulations on the shown set S of five points. We study the following problem:

Question: *In how many ways can a combinatorial triangulation with n vertices be drawn on a set of n points in the plane?*

Note that any upper bound c^n on this number yields trivially an upper bound for $tr(n)$ of $O((c \cdot 9,\overline{481})^n)$. It turns out to be very difficult to find examples of combinatorial triangulations which can be drawn in many different ways on a given point set S. A first simple bound is shown in the following:

• A triangulation formed by nested triangles The set S of n points in the plane in general position has $\frac{n}{3}$ layers of three points as in Fig. 2. The combinatorial triangulation T we consider is the one shown in this figure. We observe that each triangular layer can be rotated to produce a different geometric triangulation of S, while maintaining the combinatorial triangulation T. This yields a lower bound of

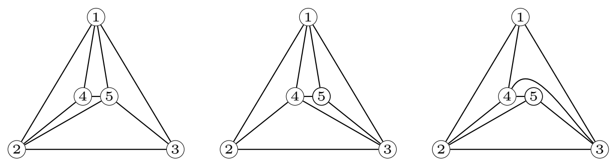

Figure 1: Combinatorial triangulations on 5 vertices.

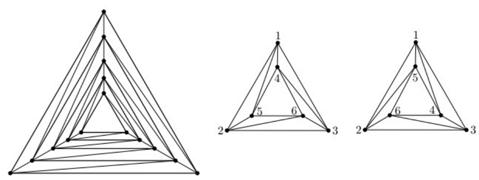

Figure 2: A triangulation formed by nested triangles and a rotation between consecutive layers.

$\Omega\left(2^{\frac{n}{3}}\right) = \Omega\left(1,2599^n\right)$ different drawings of T on S.

• A triangulation on the double chain We improve upon this bound by defining another combinatorial triangulation T recursively, and show a lower bound on the number of drawings of T on the so-called *double chain* point configuration [1].

Theorem 1 *There exists a combinatorial triangulation T and a set S of n points in the plane such that T has at least $\Omega(1,31^n)$ different drawings on S.*

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